The posted lessons are part of the Differential Equations course that I taught at Montgomery College in Germantown Maryland.

The lessons are written according to *Differential Equations* , Third Edition, by Blanchard, Devaney, and Hall , Brooks/Cole as the text book adopted for the class.

For any questions, comments or objections

You may write to me at

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Lesson 5:

Please make sure to read the sections 3.,3.5,3.6 in the text for this week

Recall that for a matrix A, λ is an eigenvalue if

$AV = \lambda V$ **for some nonzero** V

$AV - \lambda V = \mathbf{0}$

$AV - \lambda IV = \mathbf{0}$

$(A - \lambda I)\mathbf{V} = \mathbf{0}$

In order to have a nonzero V like the above, must have

$|A - \lambda I| = \mathbf{0}$

For a 2×2 **matrix** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A - \lambda I| = \mathbf{0}$

means

$$\begin{vmatrix} a - \lambda & 0 \\ 0 & b - \lambda \end{vmatrix} = \mathbf{0}$$

In the last lesson we worked with the linear systems, where the eigenvalues were both real and distinct

In this lesson, let us work with the systems, where we may have

A) nonreal eigen values

B) identical eigenvalues OR 0 as an eigenvalue

Example

1.

$\frac{dY}{dt} = \mathbf{AY}$

where $A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$

To find the eigenvalues, solve

$$\begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = \mathbf{0}$$

$(-3-\lambda)(-1-\lambda) - (2)(-1) = \mathbf{0}$

or

$(3+\lambda)(1+\lambda) + \mathbf{2} = \mathbf{0}$

or

$\lambda^2 + 4\lambda + \mathbf{3} + \mathbf{2} = \mathbf{0}$

that is

$\lambda^2 + 4\lambda + 5 = 0$

use the quadratic formula

$$\lambda = \frac{-4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2 \times 1}$$
$$\lambda = \frac{-4 \pm \sqrt{16 - 20}}{2}$$
$$\lambda = \frac{-4 \pm \sqrt{-4}}{2}$$
$$\lambda = \frac{-4 \pm 2i}{2}$$
$$\lambda = -2 \pm i$$

Eigenvectors

For

$-2 + i$

to find $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

such that

$$\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-2 + i) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{bmatrix} -3x_1 + 2x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} (-2+i)x_1 \\ (-2+i)x_2 \end{bmatrix}$$

or

$-3\mathbf{x}_1 + 2\mathbf{x}_2 = (-2+i)\mathbf{x}_1 \rightarrow 2\mathbf{x}_2 = (-2+i)\mathbf{x}_1 + 3\mathbf{x}_1 \rightarrow 2\mathbf{x}_2 = \mathbf{x}_1 + i\mathbf{x}_1 \rightarrow \mathbf{x}_2 = \frac{(1+i)}{2}\mathbf{x}_1$

$-\mathbf{x}_1 - \mathbf{x}_2 = (-2+i)\mathbf{x}_2 \rightarrow -\mathbf{x}_1 = \mathbf{x}_2 + (-2+i)\mathbf{x}_2 \rightarrow -\mathbf{x}_1 = -\mathbf{x}_2 + i\mathbf{x}_2 \rightarrow \mathbf{x}_1 = (1-i)\mathbf{x}_2 \rightarrow \mathbf{x}_2 = \frac{1}{1-i}\mathbf{x}_1$

Note that

$$\frac{(1+i)}{2} = \frac{(1+i)(1-i)}{2(1-i)} = \frac{1^2 - i^2}{2(1-i)} = \frac{1-(-1)}{2(1-i)} = \frac{2}{2(1-i)} = \frac{1}{1-i}$$

Take up $x_2 = \frac{(1+i)}{2}x_1$

a vector setup will give us

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{(1+i)}{2} \end{bmatrix} x_1$ **as an eigenvector of the matrix of the system for** $x_1 \neq 0$

Since a nonzero multiple of an eigenvector of a matrix A is again an eigenvector of A

therefore

$\begin{bmatrix} 1 \\ \frac{(1+i)}{2} \end{bmatrix}$ **is an eigenvector of** $\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$ **corresponding to the eigenvalue** $-2+i$

Therefore a solution of the equation is

remember that if v is an eigenvector corresponding to an eigenvalue λ **for a linear system,**

$e^{\lambda t}v$ **is a solution**

$$\mathbf{e}^{(-2+i)t}\begin{bmatrix} 1 \\ \frac{(1+i)}{2} \end{bmatrix} = \mathbf{e}^{-2t}\mathbf{e}^{it}\begin{bmatrix} 1 \\ \frac{(1+i)}{2} \end{bmatrix}$$

Use the important identity $\boxed{e^{iat} = \cos at + i\sin at}$ **often called the Euler's formula**

A solution is

$e^{-2t}e^{it}\begin{bmatrix} 1 \\ \frac{(1+i)}{2} \end{bmatrix}$ **Let us proceed to simplify this solution**

$$=e^{-2t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ \frac{(1+i)}{2} \end{bmatrix}$$

$$=e^{-2t}(\cos t + i\sin t)\left(\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} i\right)$$

$$=e^{-2t}\left(\cos t\begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}+\cos t\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}i+\sin t\begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}i-\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}\sin t\right)$$

$$=e^{-2t}\left(\cos t\begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}-\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}\sin t\right)+ie^{-2t}\left(\cos t\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}+\sin t\begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}\right)$$

we can have

$$e^{-2t}\left(\cos t\begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}-\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}\sin t\right)$$

$$=e^{-2t}\begin{bmatrix}\cos t\\ \frac{1}{2}(\cos t-\sin t)\end{bmatrix} \quad \text{.......... (1)}$$

as one solution

and

$$\mathbf{e}^{-2t}\left(\cos t\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}+\sin t\begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}\right)$$

$$=e^{-2t}\begin{bmatrix}\sin t\\ \frac{1}{2}(\cos t+\sin t)\end{bmatrix} \quad \text{as another solution... (2)}$$

that is we substitute these values in

$\frac{dY}{dt}=AY$

we should get an identity

Let us go ahead and verify that (1) is a solution

$$\mathbf{Y} = \mathbf{e}^{-2t}\begin{bmatrix} \cos t \\ \frac{1}{2}(\cos t - \sin t) \end{bmatrix} = \begin{bmatrix} e^{-2t}\cos t \\ \frac{1}{2}(e^{-2t}\cos t - e^{-2t}\sin t) \end{bmatrix}$$

$$\frac{dY}{dt} = \begin{bmatrix} -e^{-2t}\sin t - 2e^{-2t}\cos t \\ \frac{1}{2}(-e^{-2t}\sin t - 2e^{-2t}\cos t - e^{-2t}\cos t + 2e^{-2t}\sin t) \end{bmatrix}$$

$$\frac{dY}{dt} = \begin{bmatrix} -e^{-2t}\sin t - 2e^{-2t}\cos t \\ \frac{1}{2}(e^{-2t}\sin t - 3e^{-2t}\cos t) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

$$\mathbf{AY} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} e^{-2t}\cos t - e^{-2t}\sin t \\ \frac{1}{2}(e^{-2t}\cos t - e^{-2t}\sin t) \end{bmatrix}$$

multiply the two matrices

$$\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} e^{-2t}\cos t \\ \frac{1}{2}(e^{-2t}\cos t - e^{-2t}\sin t) \end{bmatrix} = \begin{bmatrix} -2e^{-2t}\cos t - e^{-2t}\sin t \\ -\frac{3}{2}e^{-2t}\cos t + \frac{1}{2}e^{-2t}\sin t \end{bmatrix}$$

GOOD

To check

$$Y = e^{-2t}\begin{bmatrix} \sin t \\ \frac{1}{2}(\cos t + \sin t) \end{bmatrix} = \begin{bmatrix} e^{-2t}\sin t \\ \frac{1}{2}e^{-2t}(\cos t + \sin t) \end{bmatrix}$$ **is also a solution**

of

$$\frac{dY}{dt} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \mathbf{Y}$$

$$\frac{dY}{dt} = \begin{bmatrix} -2e^{-2t}\sin t + e^{-2t}\cos t \\ \frac{1}{2}(-e^{-2t}\sin t - 2e^{-2t}\cos t + e^{-2t}\cos t - 2e^{-2t}\sin t) \end{bmatrix}$$

or

$$\frac{dY}{dt} = \begin{bmatrix} -2e^{-2t}\sin t + e^{-2t}\cos t \\ \frac{1}{2}(-e^{-2t}\cos t - 3e^{-2t}\sin t) \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \mathbf{Y}$$

$$= \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} e^{-2t}\sin t \\ \frac{1}{2}e^{-2t}\cos t + \frac{1}{2}e^{-2t}\sin t \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{-2t}\sin t + e^{-2t}\cos t \\ -\frac{3}{2}e^{-2t}\sin t - \frac{1}{2}e^{-2t}\cos t \end{bmatrix}$$ **Did very good**

Now that we have two linearly independent solutions of the system, we may not really have to worry about the working with the other values,

but for this particular example, let us go ahead and look at that any ways

You may see that $\begin{bmatrix} 1 \\ \frac{1}{2} - \frac{1}{2}i \end{bmatrix}$ **is an eigen vector of A**

corresponding to $-2-i$

$\begin{bmatrix} 1 \\ \frac{1}{2} - \frac{1}{2}i \end{bmatrix}$ **is an eigenvector**

$2\begin{bmatrix} 1 \\ \frac{1}{2} - \frac{1}{2}i \end{bmatrix} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$ **is also an vector corresponding to** $(-2-i)$

$e^{(-2-i)t}\begin{bmatrix} 2 \\ 1-i \end{bmatrix}$ **is a solution**

(we are wondering about it)

$\mathbf{e}^{(-2-i)t}\begin{bmatrix} 2 \\ 1-i \end{bmatrix}$

$= \mathbf{e}^{-2t}\mathbf{e}^{i(-t)}\begin{bmatrix} 2 \\ 1-i \end{bmatrix}$

$=e^{-2t}(\cos(-t) + i\sin(-t))\begin{bmatrix} 2 \\ 1-i \end{bmatrix}$

$=e^{-2t}(\cos t - i\sin t)\begin{bmatrix} 2 \\ 1-i \end{bmatrix}$

$$=e^{-2t}(\cos t - i\sin t)\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} i\right)$$

$$=e^{-2t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos t + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t - i\begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right)$$

$$=e^{-2t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right) + e^{-2t}\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin t\right)i$$

$$=\begin{bmatrix} 2e^{-2t}\cos t \\ e^{-2t}(\cos t - \sin t) \end{bmatrix} + \begin{bmatrix} -2e^{-2t}\sin t \\ -e^{-2t}(\cos t + \sin t) \end{bmatrix} i$$

$\begin{bmatrix} 2e^{-2t}\cos t \\ e^{-2t}(\cos t - \sin t) \end{bmatrix}$ **a solution**

$\begin{bmatrix} -2e^{-2t}\sin t \\ -e^{-2t}(\cos t + \sin t) \end{bmatrix}$ **is another solution**

HAD

$e^{-2t}\begin{bmatrix} \cos t \\ \frac{1}{2}(\cos t - \sin t) \end{bmatrix}$ **.......... (1)**

as one solution
and

$=e^{-2t}\begin{bmatrix} \sin t \\ \frac{1}{2}(\cos t + \sin t) \end{bmatrix}$ **as another solution... (2)**

For the imginary eigenvalues, once we find the solution using the eigenvalue $\alpha + i\beta$ we do not have to work with $\alpha - i\beta$

Again, our equation is

$\frac{dY}{dt} =$ **AY**

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

One solution is

$$\mathbf{Y}_1 = \mathbf{e}^{-2t} \begin{bmatrix} \cos t \\ \frac{1}{2}(\cos t - \sin t) \end{bmatrix} = \begin{bmatrix} e^{-2t} \cos t \\ \frac{e^{-2t}}{2}(\cos t - \sin t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

and another solution is

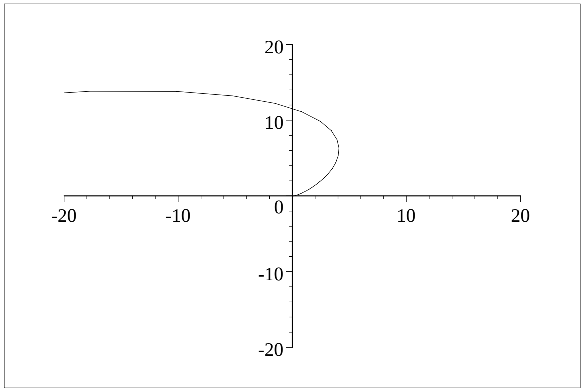
$$\mathbf{Y}_2(t) = \begin{bmatrix} e^{-2t} \sin t \\ \frac{1}{2} e^{-2t}(\cos t + \sin t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

We can graph these values as parametric curves in the phase plane

FOR

$$\mathbf{Y}_1 = \begin{bmatrix} e^{-2t} \cos t \\ \frac{e^{-2t}}{2}(\cos t - \sin t) \end{bmatrix}$$

a graph is

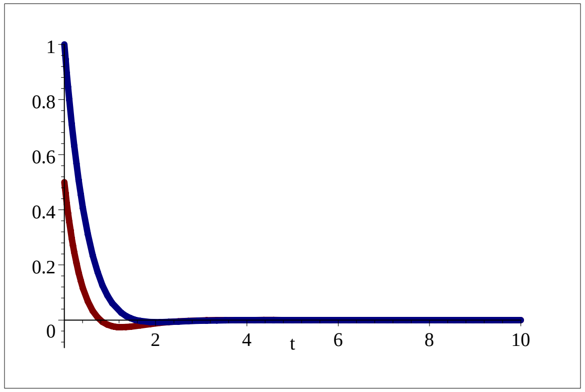


Graphs of

$x(t) = e^{-2t}\cos t$ **BLUE**
$y(t) = \frac{e^{-2t}}{2}(\cos t - \sin t)$ **RED**

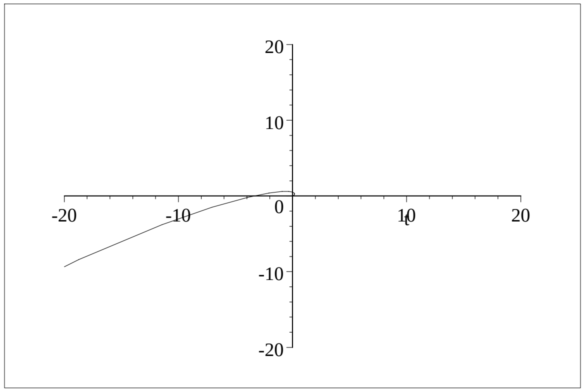
are

$e^{-2t}\cos t$

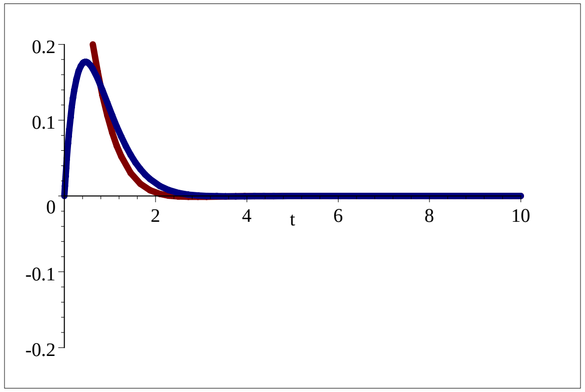


For

$$Y_2(t) = \begin{bmatrix} e^{-2t}\sin t \\ \frac{1}{2}e^{-2t}(\cos t + \sin t) \end{bmatrix}$$



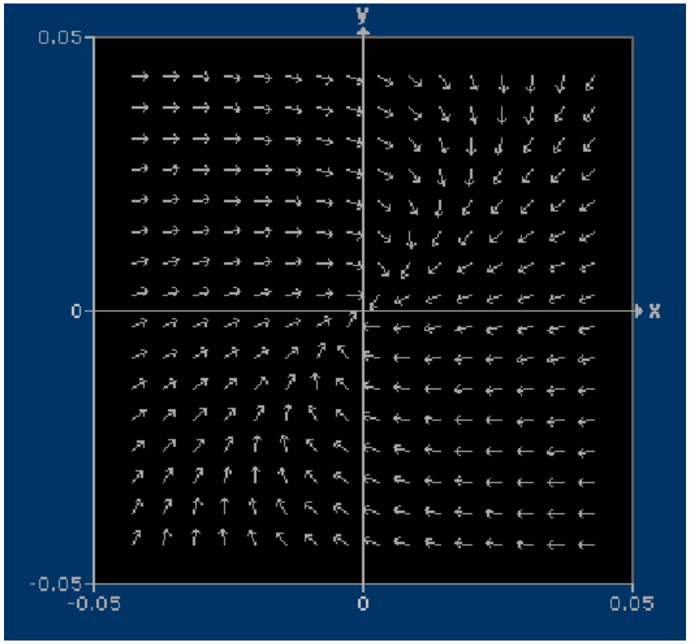
$e^{-2t}\sin t$



GENERAL SOLUTION FOR THIS SYSTEM IS

$$\mathbf{Y} = \mathbf{c}_1 \begin{bmatrix} e^{-2t}\cos t \\ \frac{e^{-2t}}{2}(\cos t - \sin t) \end{bmatrix} + \mathbf{c}_2 \begin{bmatrix} e^{-2t}\sin t \\ \frac{1}{2}e^{-2t}(\cos t + \sin t) \end{bmatrix}$$

Let us use HPG system solver to see the vector fields



Note that the solutions spiral towards the origin because of e^{-2t}

Example 2:

To find the solutions of

$\frac{dY}{dt} =$ **AY**

where $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$**, eigenvalues** $1 \pm 2i$

$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$**, eigenvectors:** $\left\{\begin{bmatrix} 1 \\ 1-i \end{bmatrix}\right\} \leftrightarrow 1+2i, \left\{\begin{bmatrix} 1 \\ 1+i \end{bmatrix}\right\} \leftrightarrow 1-2i$

$\left\{\begin{bmatrix} 1 \\ 1-i \end{bmatrix}\right\} \leftrightarrow \mathbf{1} + \mathbf{2i}$

gives

$$\mathbf{e}^{(1+2i)t}\begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \mathbf{e}^{t}\mathbf{e}^{2ti}\begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \mathbf{e}^{t}(\cos 2t + i\sin 2t)\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

$$\mathbf{e}^{t}(\cos 2t + i\sin 2t)\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$$= e^{t}(\cos 2t + i\sin 2t)\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} i\right)$$

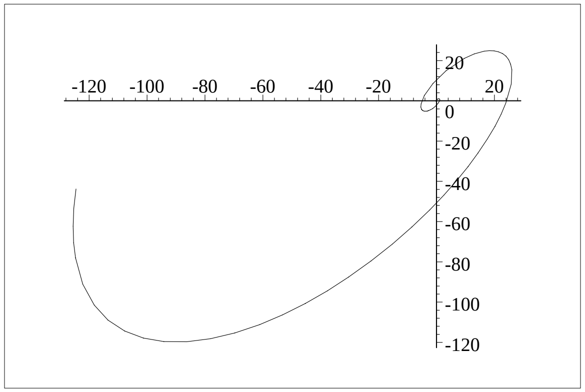
$$= e^{t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos 2t + i\sin 2t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin 2t\right)$$

$$= e^{t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin 2t\right) + e^{t}\left(-\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos 2t + \sin 2t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)i$$

$$= e^{t}\begin{bmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{bmatrix} + e^{t}\begin{bmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{bmatrix} i$$

$e^{t}\begin{bmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{bmatrix} = \begin{bmatrix} e^{t}\cos 2t \\ e^{t}\cos 2t - e^{t}\sin 2t \end{bmatrix}$ **is a solution**

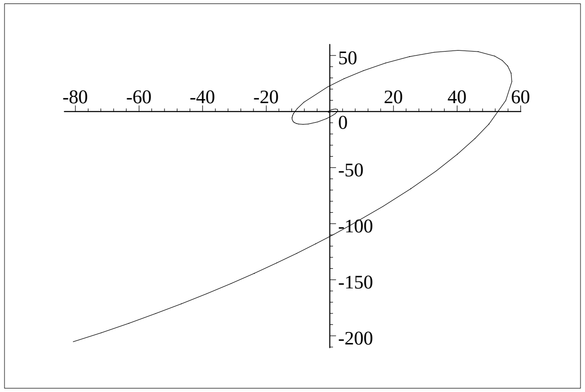
a graph of the solution is shown below



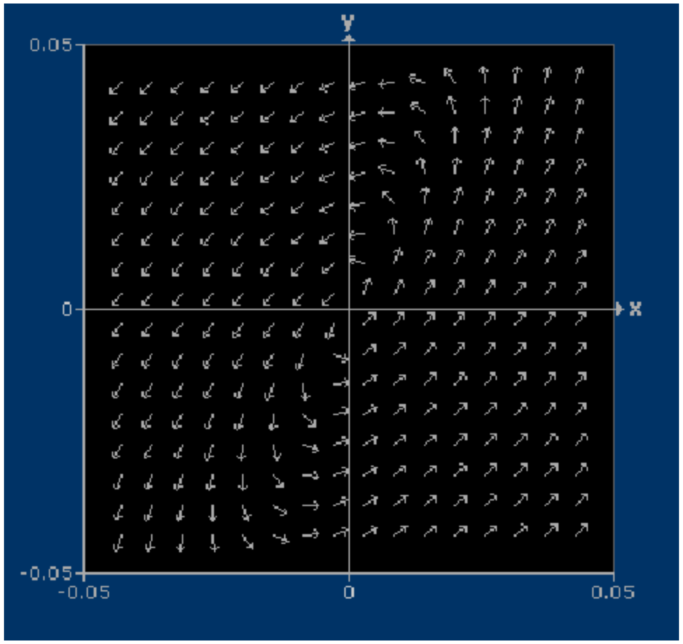
$$\mathbf{e}^{t}\begin{bmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{bmatrix}$$

$$= \begin{bmatrix} e^{t}\sin 2t \\ e^{t}\cos 2t + e^{t}\sin 2t \end{bmatrix}$$

$$= \begin{bmatrix} e^{t}\sin 2t \\ e^{t}\cos 2t + e^{t}\sin 2t \end{bmatrix}$$ **a graph is shown below**



If we use the HPG systemsolver, then



You can see that the solutions spiral away from the origin.

MUST READ: The discussion on the pages 297-300

6. on the page 305

For the linear system

$$\frac{dY}{dt} = \begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} \mathbf{Y}$$

a) to find the eigenvalues of A

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix}$$

For the eigenvalues, we have to solve the charecterstic equation

$$\begin{vmatrix} 0-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = \mathbf{0}$$

that is

$-\boldsymbol{\lambda}(-1-\lambda)-(-2)(2)= \mathbf{0}$

$\boldsymbol{\lambda}(1+\lambda)+\mathbf{4} = \mathbf{0}$

$\boldsymbol{\lambda} + \boldsymbol{\lambda}^2+\mathbf{4} = \mathbf{0}$

or

$\boldsymbol{\lambda}^2+\boldsymbol{\lambda} + \mathbf{4} = \mathbf{0}$

use the quadratic formula to obtain

$\lambda = \frac{-1 \pm \sqrt{1-16}}{2}$
or
$\lambda = \frac{-1 \pm \sqrt{15}\, i}{2}$

b) to determine if the origin is a spiral source or spiral sink or a center

Since the real part of the Eigenvalues is negative, the origin is a spiral sink.

c) to determine the natural period and the natural frequency of the oscillations

Remember that if $\alpha \pm \beta i$ are the eigenvalues, the natural period is $\frac{2\pi}{\beta}$ and the natural frequency is $\frac{\beta}{2\pi}$ because of the nature of $\sin \beta t$ and $\cos \beta t$

In this case $\beta = \frac{\sqrt{15}}{2}$

Natural period is $\frac{2\pi}{\left(\frac{\sqrt{15}}{2}\right)} = \frac{4\pi}{\sqrt{15}}$

Natural Frequency is $\frac{\sqrt{15}}{4\pi}$

d) determine the direction of the oscillations in the phase plane that is to say to determine whether the solutions go clockwise or anticlockwise around the origin

Remember (I hope that you read the portions I requested) that eventhough this question is best answered by looking at a direction field, we can get an idea just by looking at the direction of one nonzer vector of the

direction field. A good startegy is to look at the direction field at $(1,0)$,

if it points up in the first quadrant, solutions spiral in the counter clockwise direction

if it points down in the fourth quadrant, solutions spiral in the clockwise direction

$$\frac{dY}{dt} = \begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} Y$$

at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

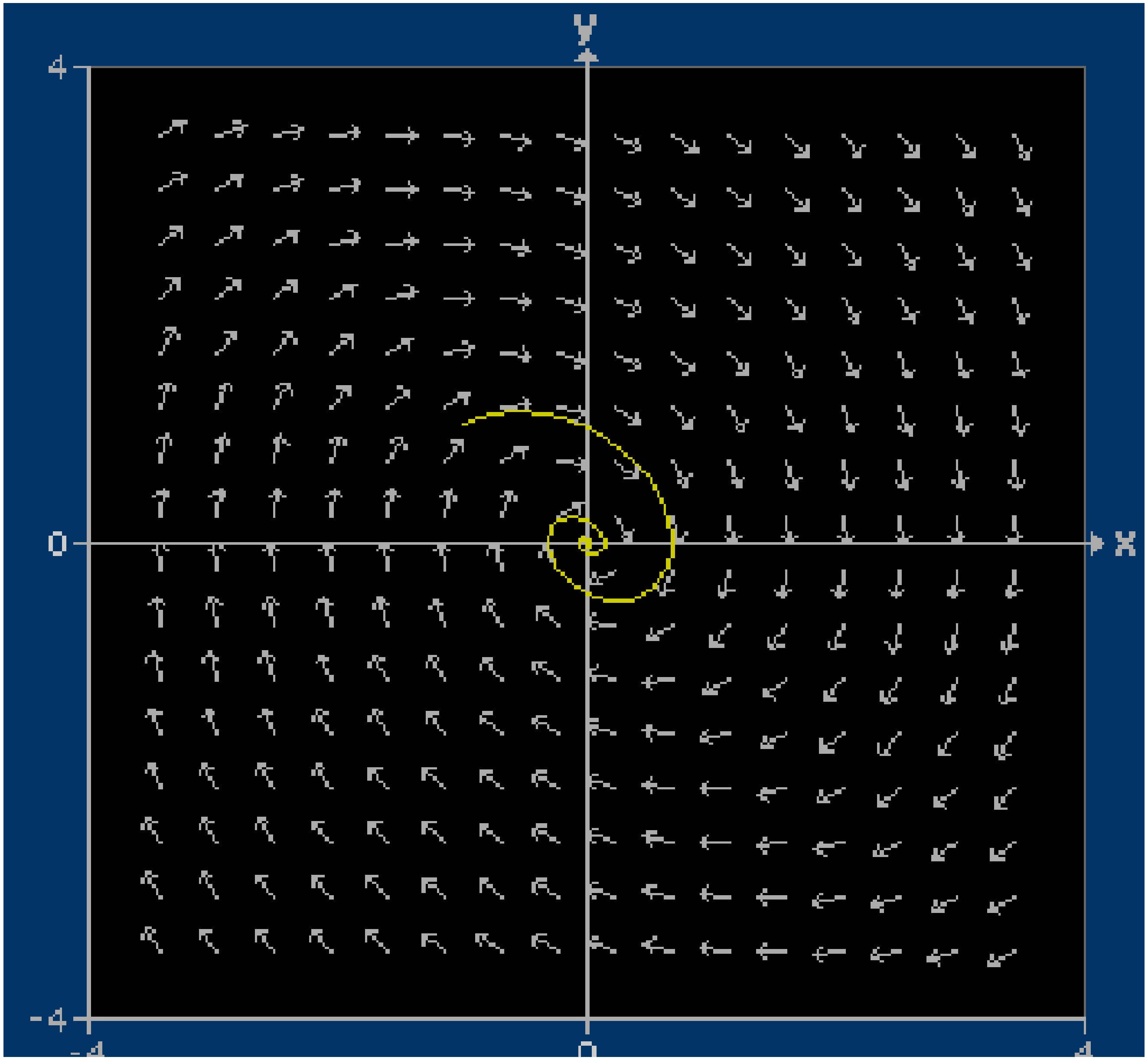
this vector points downwards, therefore the solutions spiral in the clockwise direction around the origin.

e)

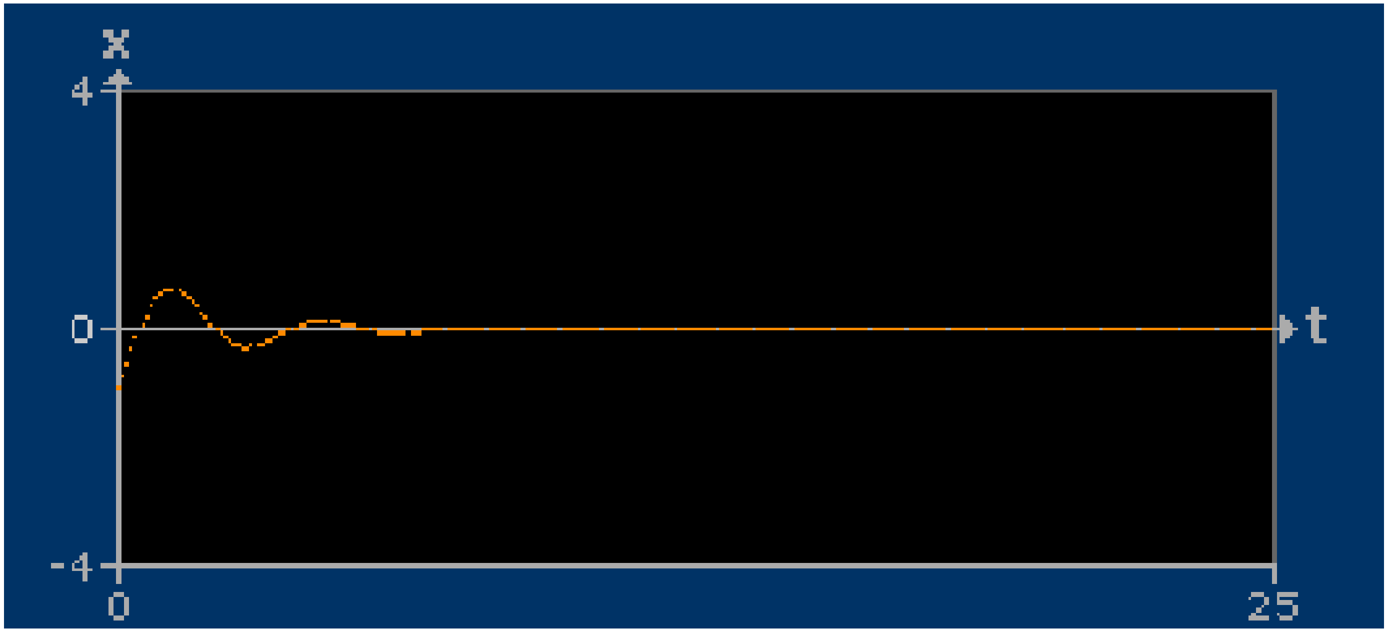
To use the HPG systemsolver to sketch the xy-phase portrait and the $x(t)$ and $y(t)$

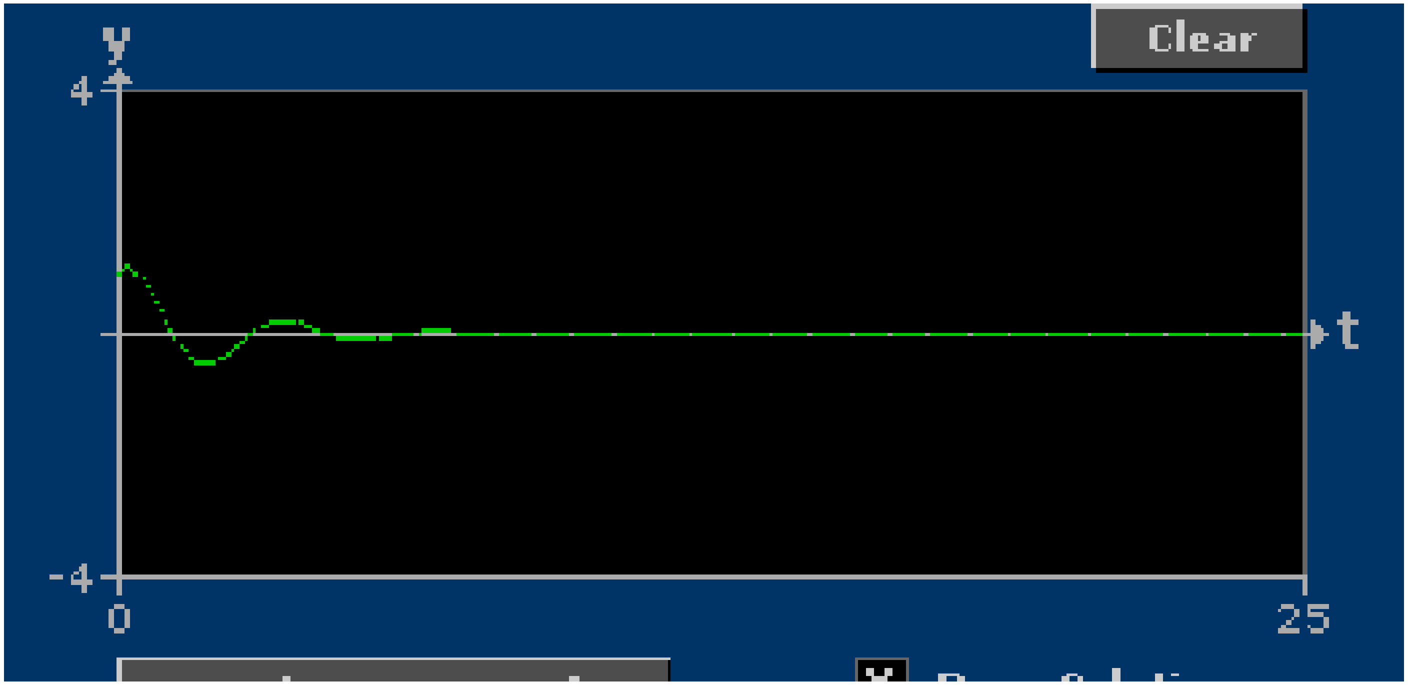
graphs corresponding to the initial condition $Y_0 = (-1,1)$

the computer output is



y
x
4
0
-4
-4
0
4


x
4
0
-4
0
25
t



#12 page 306

Here we have to work the same problem (#6 on the page 305) out by using the eigenvectors

For the eigenvectors

Corresponding to

$\lambda = \frac{-1-\sqrt{15}\,i}{2}$

solve

$$\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{-1-\sqrt{15}\,i}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

to get

$$\begin{bmatrix} 2x_2 \\ -2x_1 - x_2 \end{bmatrix} = \begin{bmatrix} \frac{-1-\sqrt{15}\,i}{2}x_1 \\ \frac{-1-\sqrt{15}\,i}{2}x_2 \end{bmatrix}$$

that is

$2x_2 = \frac{-1-\sqrt{15}\,i}{2}x_1 \rightarrow x_2 = \frac{-1-\sqrt{15}\,i}{4}x_1$

$-2x_1 - x_2 = \frac{-1-\sqrt{15}\,i}{2}x_2 \rightarrow -2x_1 = \frac{-1-\sqrt{15}\,i}{2}x_2 + x_2 \rightarrow -2x_1 = \frac{1-\sqrt{15}\,i}{2}x_2 \rightarrow x_2 = -\frac{4}{1-\sqrt{15}\,i}x_1$

verify that

$\frac{-1-\sqrt{15}\,i}{4} = -\frac{4}{1-\sqrt{15}\,i}$ **is true**

therefore an eigenvector of $\lambda = \frac{-1-\sqrt{15}\,i}{2}$ **is**

$$\begin{bmatrix} 1 \\ \frac{-1-\sqrt{15}\,i}{4} \end{bmatrix}$$

therefore a solution corresponding to

$\lambda = \frac{-1-\sqrt{15}\,i}{2}$ **may be written as**

$$\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix}, \textbf{ eigenvectors: } \left\{\begin{bmatrix} 1 \\ -\frac{1}{4} + \frac{1}{4}i\sqrt{15} \end{bmatrix}\right\} \leftrightarrow -\frac{1}{2} + \frac{1}{2}i\sqrt{15}, \left\{\begin{bmatrix} 1 \\ -\frac{1}{4} - \frac{1}{4}i\sqrt{15} \end{bmatrix}\right\} \leftrightarrow -\frac{1}{2} - \frac{1}{2}i\sqrt{15}$$

We get a complex solution

$$\mathbf{Y}(t) = \mathbf{e}^{\left(-\frac{1}{2}+\frac{1}{2}i\sqrt{15}\right)t} \begin{bmatrix} 1 \\ -\frac{1}{4} + \frac{1}{4}i\sqrt{15} \end{bmatrix}$$

that is

$$\mathbf{Y}(t) = \mathbf{e}^{-(1/2)t}\mathbf{e}^{\left(\frac{1}{2}i\sqrt{15}\right)t} \begin{bmatrix} 1 \\ -\frac{1}{4} + \frac{1}{4}i\sqrt{15} \end{bmatrix}$$

remember $e^{\left(\frac{1}{2}i\sqrt{15}\right)t} = \cos\left(\frac{\sqrt{15}}{2}\right)t + i\sin\left(\frac{\sqrt{15}}{2}\right)t$

$$\mathbf{Y}(t) = \mathbf{e}^{-(1/2)t}\mathbf{e}^{\left(\frac{1}{2}i\sqrt{15}\right)t} \begin{bmatrix} 1 \\ -\frac{1}{4} + \frac{1}{4}i\sqrt{15} \end{bmatrix}$$

$$=e^{-(1/2)t}\left(\cos\left(\frac{\sqrt{15}}{2}\right)t+i\sin\left(\frac{\sqrt{15}}{2}\right)t\right)\begin{bmatrix}1\\ -\frac{1}{4}+\frac{1}{4}i\sqrt{15}\end{bmatrix}$$

$$=e^{-(1/2)t}\begin{bmatrix}\left(\cos\left(\frac{\sqrt{15}}{2}\right)t+i\sin\left(\frac{\sqrt{15}}{2}\right)t\right)\\ \left(\cos\left(\frac{\sqrt{15}}{2}\right)t+i\sin\left(\frac{\sqrt{15}}{2}\right)t\right)\left(-\frac{1}{4}+\frac{1}{4}i\sqrt{15}\right)\end{bmatrix}$$

$$=e^{-(1/2)t}\begin{bmatrix}\left(\cos\left(\frac{\sqrt{15}}{2}\right)t+i\sin\left(\frac{\sqrt{15}}{2}\right)t\right)\\ -\frac{1}{4}\cos\left(\frac{\sqrt{15}}{2}\right)t-\frac{\sqrt{15}}{4}\sin\left(\frac{\sqrt{15}}{2}\right)t+i\left(\frac{\sqrt{15}}{4}\cos\left(\frac{\sqrt{15}}{2}\right)t-\frac{1}{4}\sin\left(\frac{\sqrt{15}}{2}\right)t\right)\end{bmatrix}$$

$$=e^{-(1/2)t}\begin{bmatrix}\cos\left(\frac{\sqrt{15}}{2}\right)t\\ -\frac{1}{4}\cos\left(\frac{\sqrt{15}}{2}\right)t-\frac{\sqrt{15}}{4}\sin\left(\frac{\sqrt{15}}{2}\right)t\end{bmatrix}+ie^{-(1/2)t}\begin{bmatrix}\sin\left(\frac{\sqrt{15}}{2}\right)t\\ \frac{\sqrt{15}}{4}\cos\left(\frac{\sqrt{15}}{2}\right)t-\frac{1}{4}\sin\left(\frac{\sqrt{15}}{2}\right)t\end{bmatrix}$$

we have linearly indepedent solutions,

$Y_1(t)=e^{-(1/2)t}\begin{bmatrix}\cos\left(\frac{\sqrt{15}}{2}\right)t\\ -\frac{1}{4}\cos\left(\frac{\sqrt{15}}{2}\right)t-\frac{\sqrt{15}}{4}\sin\left(\frac{\sqrt{15}}{2}\right)t\end{bmatrix}$ **and** $Y_2(t)=e^{-(1/2)t}\begin{bmatrix}\sin\left(\frac{\sqrt{15}}{2}\right)t\\ \frac{\sqrt{15}}{4}\cos\left(\frac{\sqrt{15}}{2}\right)t-\frac{1}{4}\sin\left(\frac{\sqrt{15}}{2}\right)t\end{bmatrix}$

giving us a general solution

$$\mathbf{Y}(t)=\lambda\mathbf{e}^{-(1/2)t}\begin{bmatrix}\cos\left(\frac{\sqrt{15}}{2}\right)t\\ -\frac{1}{4}\cos\left(\frac{\sqrt{15}}{2}\right)t-\frac{\sqrt{15}}{4}\sin\left(\frac{\sqrt{15}}{2}\right)t\end{bmatrix}+\mu\mathbf{e}^{-(1/2)t}\begin{bmatrix}\sin\left(\frac{\sqrt{15}}{2}\right)t\\ \frac{\sqrt{15}}{4}\cos\left(\frac{\sqrt{15}}{2}\right)t-\frac{1}{4}\sin\left(\frac{\sqrt{15}}{2}\right)t\end{bmatrix}$$

$$\mathbf{Y}(0)= \boldsymbol{\lambda}\mathbf{e}^{-(1/2)(0)}\begin{bmatrix} \cos\left(\frac{\sqrt{15}}{2}\right)(0) \\ -\frac{1}{4}\cos\left(\frac{\sqrt{15}}{2}\right)(0) - \frac{\sqrt{15}}{4}\sin\left(\frac{\sqrt{15}}{2}\right)(0) \end{bmatrix}+\boldsymbol{\mu}\mathbf{e}^{-(1/2)(0)}\begin{bmatrix} \sin\left(\frac{\sqrt{15}}{2}\right)(0) \\ \frac{\sqrt{15}}{4}\cos\left(\frac{\sqrt{15}}{2}\right)(0) - \frac{1}{4}\sin\left(\frac{\sqrt{15}}{2}\right)(0) \end{bmatrix}$$

$\rightarrow$

$$\mathbf{Y}_0= \boldsymbol{\lambda}\begin{bmatrix} 1 \\ -\frac{1}{4} \end{bmatrix}+\boldsymbol{\mu}\begin{bmatrix} 0 \\ \frac{\sqrt{15}}{4} \end{bmatrix}$$

$\rightarrow$

$$\mathbf{Y}_0=\begin{bmatrix} \lambda \\ \frac{-\lambda+\mu\sqrt{15}}{4} \end{bmatrix}$$

according to the initial condition,

$$\mathbf{Y}_0=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

therefore $\lambda = -1$ **,** $\frac{-\lambda+\mu\sqrt{15}}{4} = 1 \rightarrow \frac{1+\mu\sqrt{15}}{4} = 1 \rightarrow 1+\mu\sqrt{15} = 4 \rightarrow \mu\sqrt{15} = 3 \rightarrow \mu = \frac{3}{\sqrt{15}}$

The particular solution is

$$\mathbf{Y}(t)= -\mathbf{e}^{-(t/2)}\begin{bmatrix} \cos\left(\frac{\sqrt{15}}{2}t\right) \\ -\frac{1}{4}\cos\left(\frac{\sqrt{15}}{2}t\right) - \frac{\sqrt{15}}{4}\sin\left(\frac{\sqrt{15}}{2}t\right) \end{bmatrix}+\frac{3}{\sqrt{15}}\mathbf{e}^{-(t/2)}\begin{bmatrix} \sin\left(\frac{\sqrt{15}}{2}t\right) \\ \frac{\sqrt{15}}{4}\cos\left(\frac{\sqrt{15}}{2}t\right) - \frac{1}{4}\sin\left(\frac{\sqrt{15}}{2}t\right) \end{bmatrix}$$

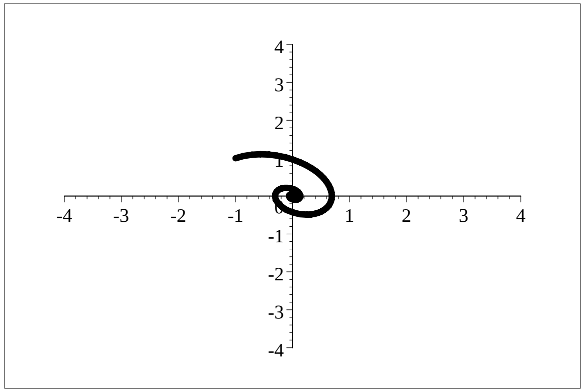
$$\mathbf{Y}(t)=\begin{bmatrix} -e^{-\frac{1}{2}t}\cos\frac{1}{2}\sqrt{15}\,t+\frac{1}{5}\sqrt{15}\,e^{-\frac{1}{2}t}\sin\frac{1}{2}\sqrt{15}\,t \\ -e^{-\frac{1}{2}t}\left(-\frac{1}{4}\cos\frac{1}{2}\sqrt{15}\,t-\frac{1}{4}\sqrt{15}\sin\frac{1}{2}\sqrt{15}\,t\right)+\frac{1}{5}\sqrt{15}\,e^{-\frac{1}{2}t}\left(\frac{1}{4}\sqrt{15}\cos\frac{1}{2}\sqrt{15}\,t-\frac{1}{4}\sin\frac{1}{2}\sqrt{15}\,t\right) \end{bmatrix}$$

$$Y(t) = e^{-\frac{1}{2}t}\begin{bmatrix} -\cos\frac{1}{2}\sqrt{15}\,t+\frac{1}{5}\sqrt{15}\sin\frac{1}{2}\sqrt{15}\,t \\ \left(\frac{1}{4}\cos\frac{1}{2}\sqrt{15}\,t+\frac{1}{4}\sqrt{15}\sin\frac{1}{2}\sqrt{15}\,t\right)+\frac{1}{5}\sqrt{15}\left(\frac{1}{4}\sqrt{15}\cos\frac{1}{2}\sqrt{15}\,t-\frac{1}{4}\sin\frac{1}{2}\sqrt{15}\,t\right) \end{bmatrix}$$

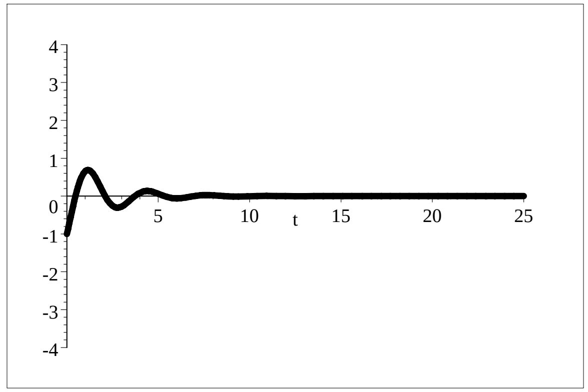
$$\mathbf{Y}(t)=\mathbf{e}^{-\frac{1}{2}t}\begin{bmatrix} -\cos\frac{1}{2}\sqrt{15}\,t+\frac{1}{5}\sqrt{15}\sin\frac{1}{2}\sqrt{15}\,t \\ \frac{1}{4}\cos\frac{1}{2}\sqrt{15}\,t+\frac{1}{4}\sqrt{15}\sin\frac{1}{2}\sqrt{15}\,t+\frac{3}{4}\cos\frac{1}{2}\sqrt{15}\,t-\frac{\sqrt{15}}{20}\sin\frac{1}{2}\sqrt{15}\,t \end{bmatrix}$$

$$\mathbf{Y}(t)=\mathbf{e}^{-\frac{1}{2}t}\begin{bmatrix} -\cos\frac{1}{2}\sqrt{15}\,t+\frac{1}{5}\sqrt{15}\sin\frac{1}{2}\sqrt{15}\,t \\ \cos\frac{1}{2}\sqrt{15}\,t+\frac{\sqrt{15}}{5}\sin\frac{1}{2}\sqrt{15}\,t \end{bmatrix}$$

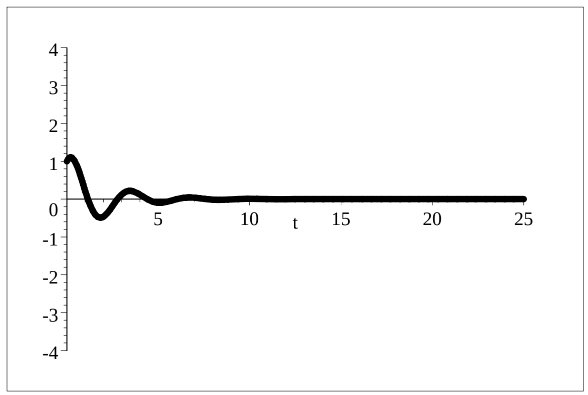
Phase portrait



$$x(t) = e^{-\frac{1}{2}t}\left(-\cos\frac{1}{2}\sqrt{15}\,t + \frac{1}{5}\sqrt{15}\,\sin\frac{1}{2}\sqrt{15}\,t\right)$$



$$y(t) = e^{-\frac{1}{2}t}\left(\cos\frac{1}{2}\sqrt{15}\,t + \frac{\sqrt{15}}{5}\,\sin\frac{1}{2}\sqrt{15}\,t\right)$$



Let us takeup an example of identical eigenvalues:

#6 on the page 321

$$\frac{dY}{dt} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \mathbf{Y}$$

To find the eigenvalues

Solve

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = \mathbf{0}$$

$(2-\lambda)(4-\lambda)+\mathbf{1} = \mathbf{0}$

$\boldsymbol{\lambda}^2 - \mathbf{6}\boldsymbol{\lambda} + \mathbf{9} = \mathbf{0}$

$(\lambda - 3)^2 = \mathbf{0}$

eigenvalues are $3, 3$ **repeated eigenvalues**

b)

Eigenvector

Look for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **such that**

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{3} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$\mathbf{2x}_1 + \mathbf{x}_2 = \mathbf{3x}_1 \rightarrow \mathbf{x}_2 = \mathbf{x}_1$

$-\mathbf{x}_1 + \mathbf{4x}_2 = \mathbf{3x}_2 \rightarrow \mathbf{x}_2 = \mathbf{x}_1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{x}_1$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **is an eigenvector corresponding to** 3

suggested solution is

$$\mathbf{e}^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

$\mathbf{x}_1 = \mathbf{x}_2$

The thoerem on the page 313 for such cases states that a general solution will be

$$\mathbf{Y}(t) = \mathbf{e}^{3t}\mathbf{V}_0 + \mathbf{t}\mathbf{e}^{3t}\mathbf{V}_1$$

where V_0 **is an initial condition and** V_1 **determined by**

$$V_1 = (A - \lambda I)V_0$$

Given the initial condition $Y_0 = (1, 0)$

$\mathbf{x}(0) = \mathbf{1}$

$\mathbf{y}(0) = \mathbf{0}$

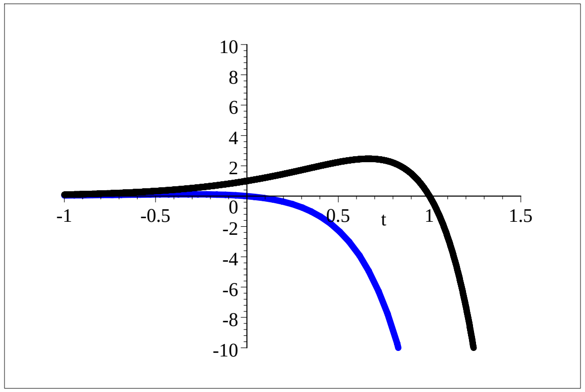
$$\mathbf{V}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{V}_1 = \left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$=\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

solution with V_0 as an initial condition

$$\mathbf{Y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathbf{e}^{3t}+\mathbf{t}\mathbf{e}^{3t}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$



x **black,** y **blue**

Let us take up Second Order Linear Equations:

Example 1:

Solve

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 0$$

If $y = e^{st}$ **, where** s **is a constant, is a solution:**

then $y = e^{st}$ **should satisfy the given differential equation**

$$\frac{dy}{dt} = se^{st}$$

$$\frac{d^2y}{dt^2} = s^2e^{st}$$

substitute in the above differential equation to obtain

$$s^2e^{st} - 3se^{st} - 4e^{st} = 0$$

that is

$$(s^2 - 3s - 4)e^{st} = 0$$

since $e^{st} \neq 0$

$$s^2 - 3s - 4 = 0$$

$$(s+1)(s-4) = 0$$

$$s = -1, 4$$

we have two linearly independent solutions

$y_1 = e^{-t}$ **and** $y_2 = e^{4t}$

and a general solution is

$y = c_1 e^{-t} + c_2 e^{4t}$

2.

$\frac{d^2y}{dt^2} + \frac{dy}{dt} + \mathbf{2y} = \mathbf{0}$

If $y = e^{st}$ **is a solution**
$\frac{dy}{dt} = \mathbf{s}\mathbf{e}^{st}$
$\frac{d^2y}{dt^2} = \mathbf{s}^2\mathbf{e}^{st}$

$\mathbf{s}^2\mathbf{e}^{st} + \mathbf{s}\mathbf{e}^{st} + \mathbf{2}\mathbf{e}^{st} = \mathbf{0}$

$s^2 + s + 2 = 0$**, Solution is:** $\left\{s = -\frac{1}{2} + \frac{1}{2}i\sqrt{7}\right\}, \left\{s = -\frac{1}{2} - \frac{1}{2}i\sqrt{7}\right\}$

General Solution is
$\mathbf{y} = \mathbf{A}\mathbf{e}^{\left(-\frac{1}{2}+\frac{1}{2}i\sqrt{7}\right)t} + \mathbf{B}\mathbf{e}^{\left(-\frac{1}{2}-\frac{1}{2}i\sqrt{7}\right)t}$

Using Euler's Formula, we may deduce that

$\mathbf{y} = \mathbf{e}^{(-1/2)t}\left(C_1 \cos\left(\frac{\sqrt{7}}{2}\right)t + C_2 \sin\left(\frac{\sqrt{7}}{2}\right)t\right)$
is general solution

10 on the page 336

$\frac{d^2y}{dt^2}$+**4**$\frac{dy}{dt}$+**20y** $=$ **0**

initial condition

$y(0) = 2$ **and** $y'(0) = -8$

$s^2 + 4s + 20 = 0$**, Solutions:** $\{s = -2 + 4i\}, \{s = -2 - 4i\}$

General solution is
y(t)= **e**$^{-2t}(C_1 \cos 4t + C_2 \sin 4t)$→ **2** $=$ **C**$_1$
y$'(t)$= **–2e**$^{-2t}(C_1 \cos 4t + C_2 \sin 4t)$+**e**$^{-2t}(-4C_1 \sin 4t + 4C_2 \cos 4t)$→ **–8** $=(-2C_1 + 4C_2)$
→ **–8** $=$ **–2**(2)+**4C**$_2$→ **–4** $=$ **4C**$_2$→ **C**$_2$= **–1**

Answer:
y(t)= **e**$^{-2t}(2 \cos 4t - \sin 4t)$

Siggested Practice:
Section 3.4: 9 thru 21 odd numbered
Section 3.5: 1 thru 21 odd numbered
Section 3.6: 1 thru 11 odd numbered