The posted lessons are part of the Differential Equations course that I taught at Montgomery College in Germantown Maryland.

The lessons are written according to *Differential Equations* , Third Edition, by Blanchard, Devaney, and Hall , Brooks/Cole as the text book adopted for the class.

For any questions, comments or objections

You may write to me at

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Lesson 4: MA282 Atul Roy

Chapter 3

In this lesson, we shall discuss the solutions of the differential equations of th

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

Such differential equations are called Linear Differential Equations with constant coefficents

note that we have two dependent variables x **and** y

the coefficients a, b, c, d **are constants**

the derivatives $\frac{dx}{dt}$ **and** $\frac{dy}{dt}$ **involve only the first powers of the dependent variables.**

We shall hope to obtain the solution curves

$x = f(t)$ x, **as a function of** t
$y = g(t)$ x, **as a function of** t

and the phase portrait $u(x,y) = 0$ **, where** u **is a function of** x **and** y **a relation between** x **and** y **that emerges out of the above set of differential equations.**

Such differential equations may be written by using the matrix notation in the following manner

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let us look at an example

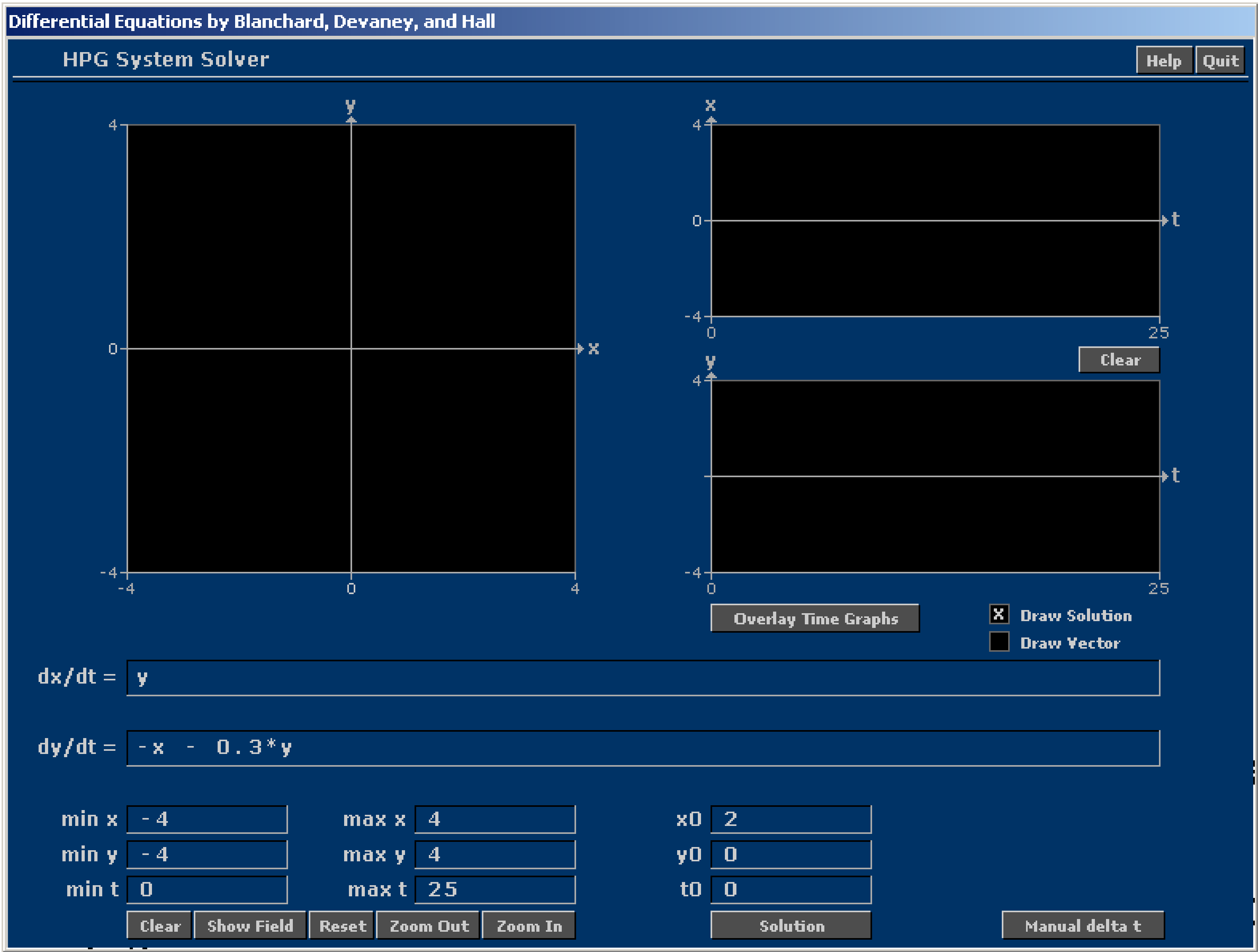
$\frac{dx}{dt} =$ **3x – 2y**
$\frac{dy}{dt} =$ **2x – 2y**

with the initial values as $x(0) = 1 \quad y(0) = -1$
May be written as

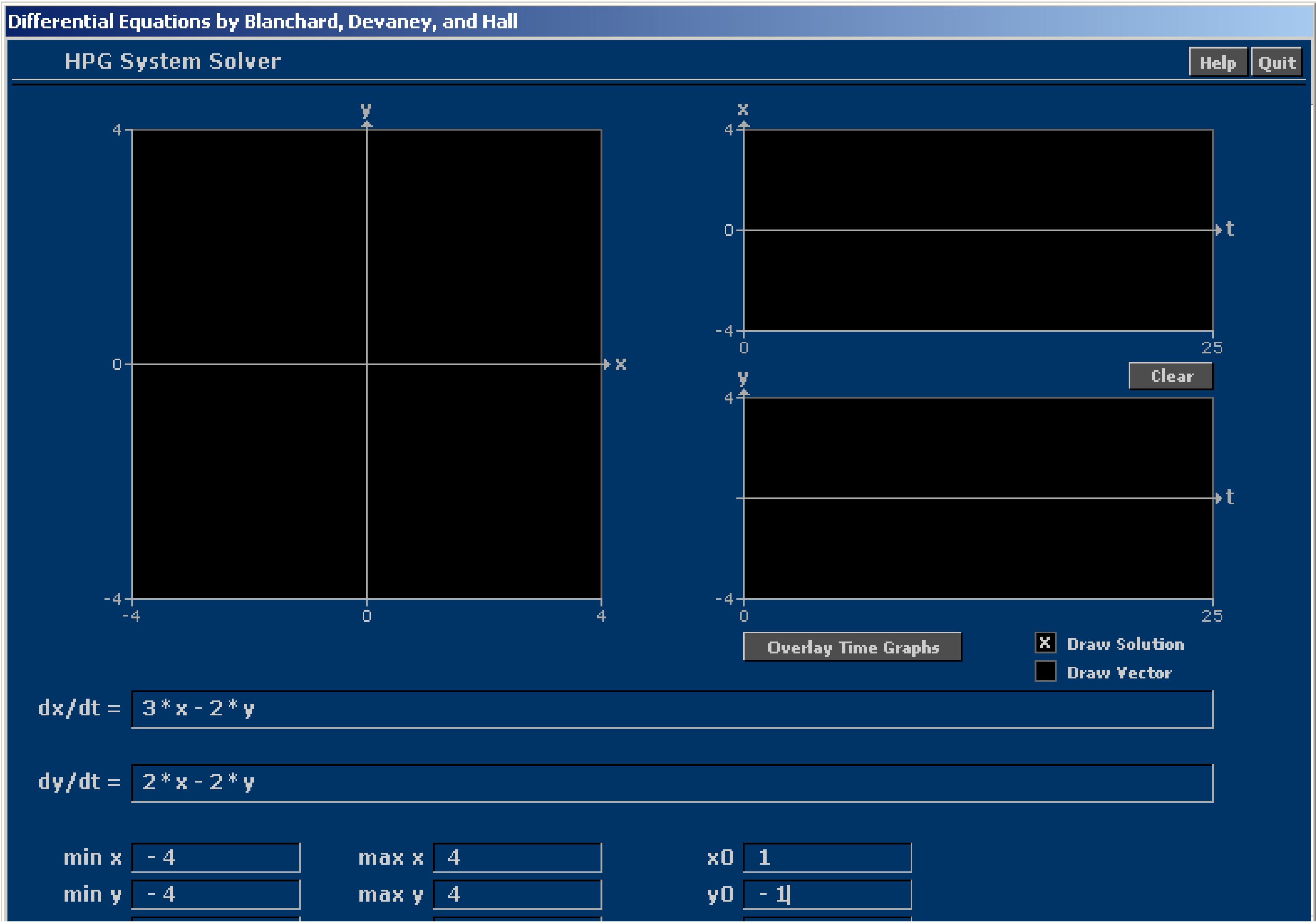
$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We have not learnt the methods of solving such equations in this course yet

Let us use the HPG system solver that has come with the CD in the text book to obtain the vector field and the solution curves. The dialogue box looks like

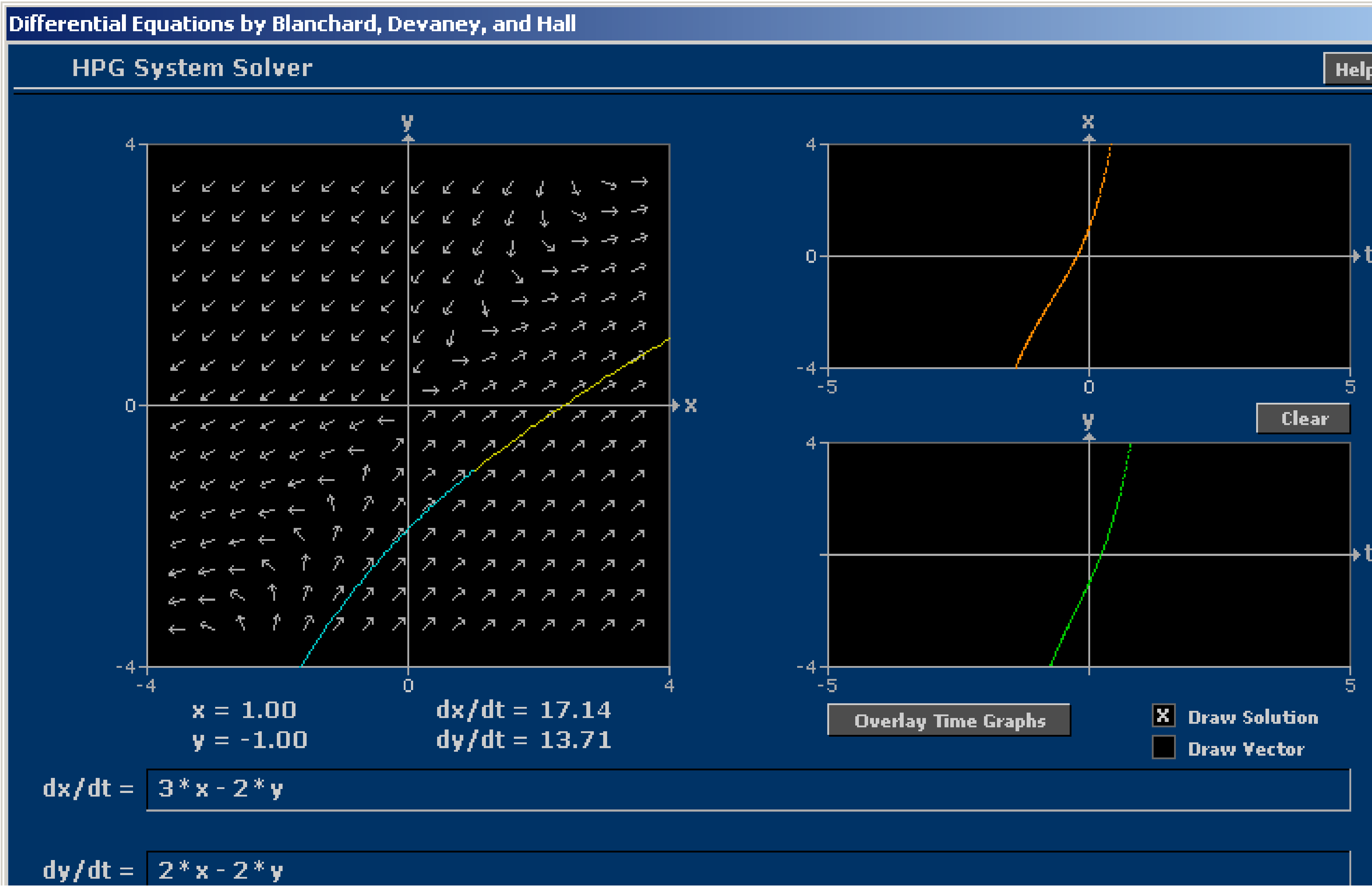

Differential Equations by Blanchard, Devaney, and Hall
HPG System Solver
Help
Quit
Clear
Overlay Time Graphs
Draw Solution
Draw Vector
dx/dt = y
dy/dt = -x - 0.3*y
min x -4
max x 4
x0 2
min y -4
max y 4
y0 0
min t 0
max t 25
t0 0
Clear
Show Field
Reset
Zoom Out
Zoom In
Solution
Manual delta t

type in the equations

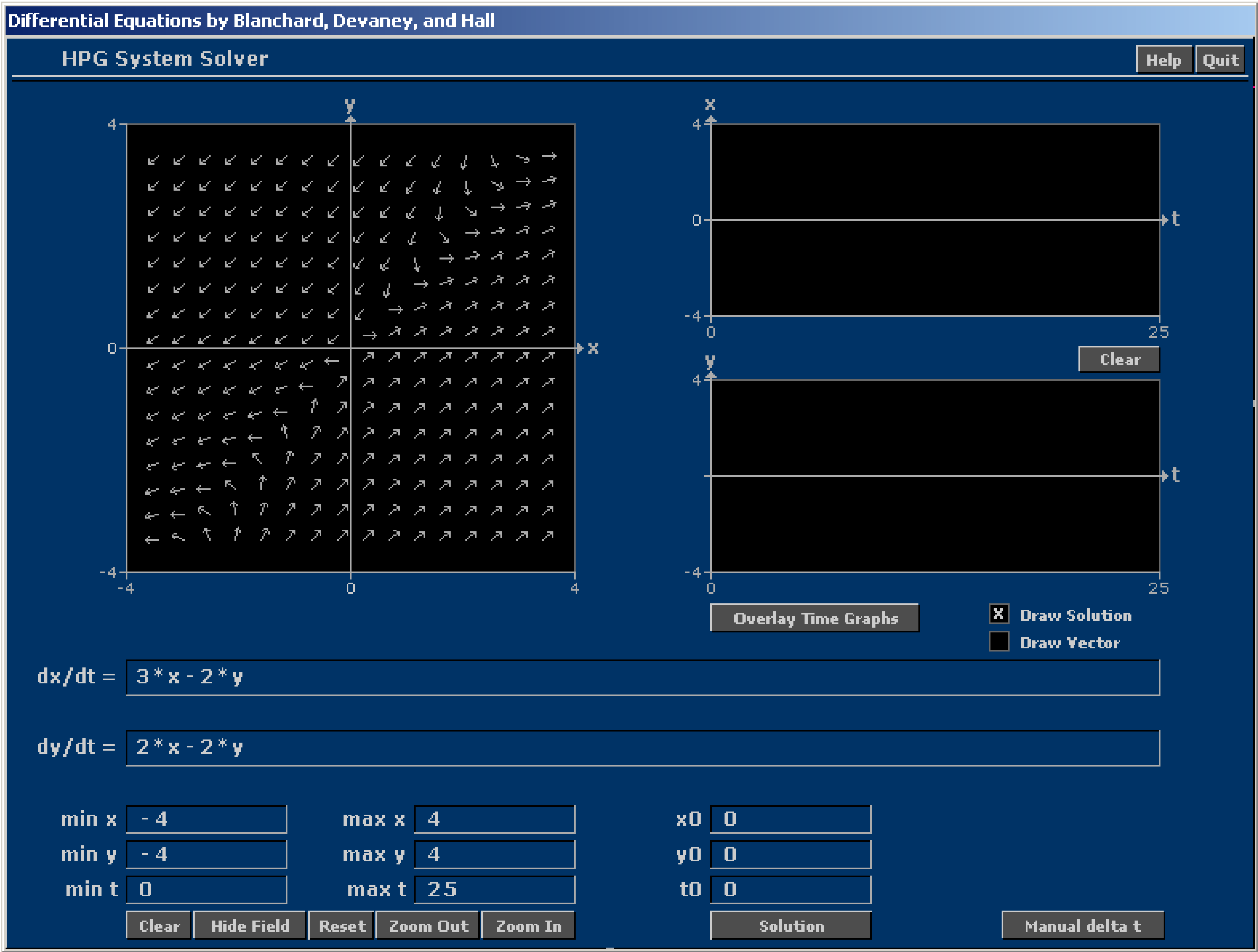

Differential Equations by Blanchard, Devaney, and Hall
HPG System Solver
Help
Quit
Clear
Overlay Time Graphs
Draw Solution
Draw Vector
dx/dt = 3*x-2*y
dy/dt = 2*x-2*y
min x -4
max x 4
x0 1
min y -4
max y 4
y0 -1

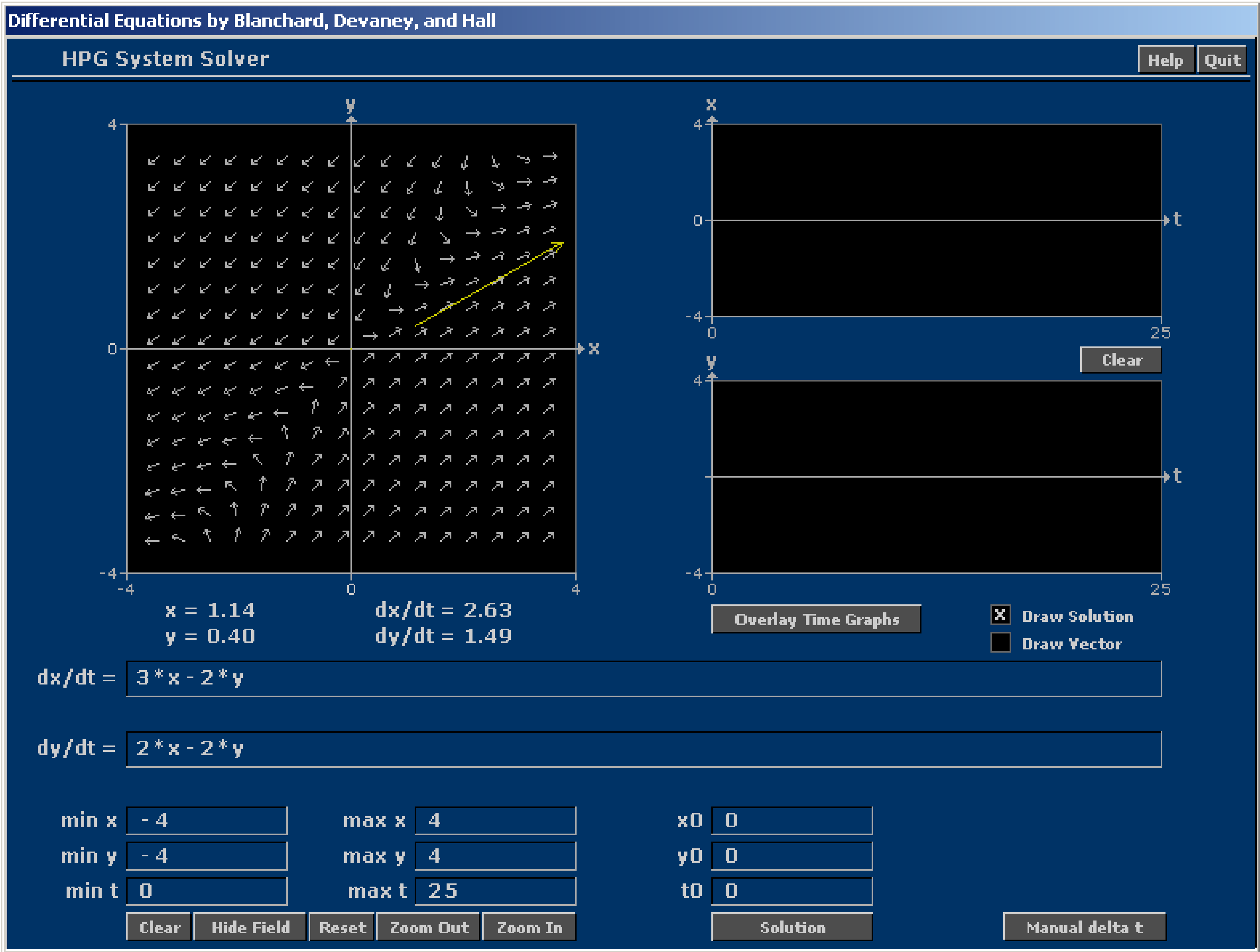
Show fields, shows you the vector field

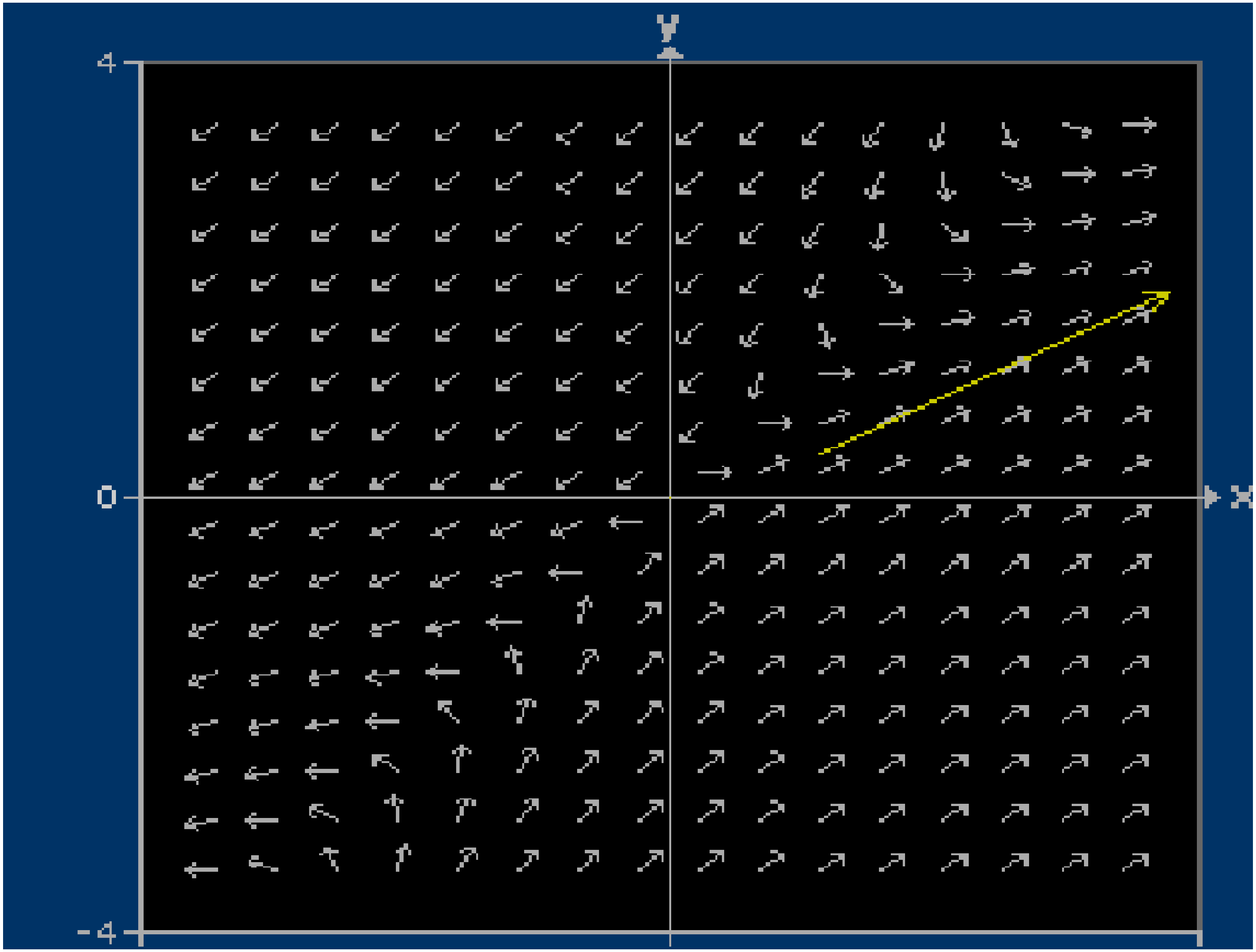
and the solutions sketches the solutions

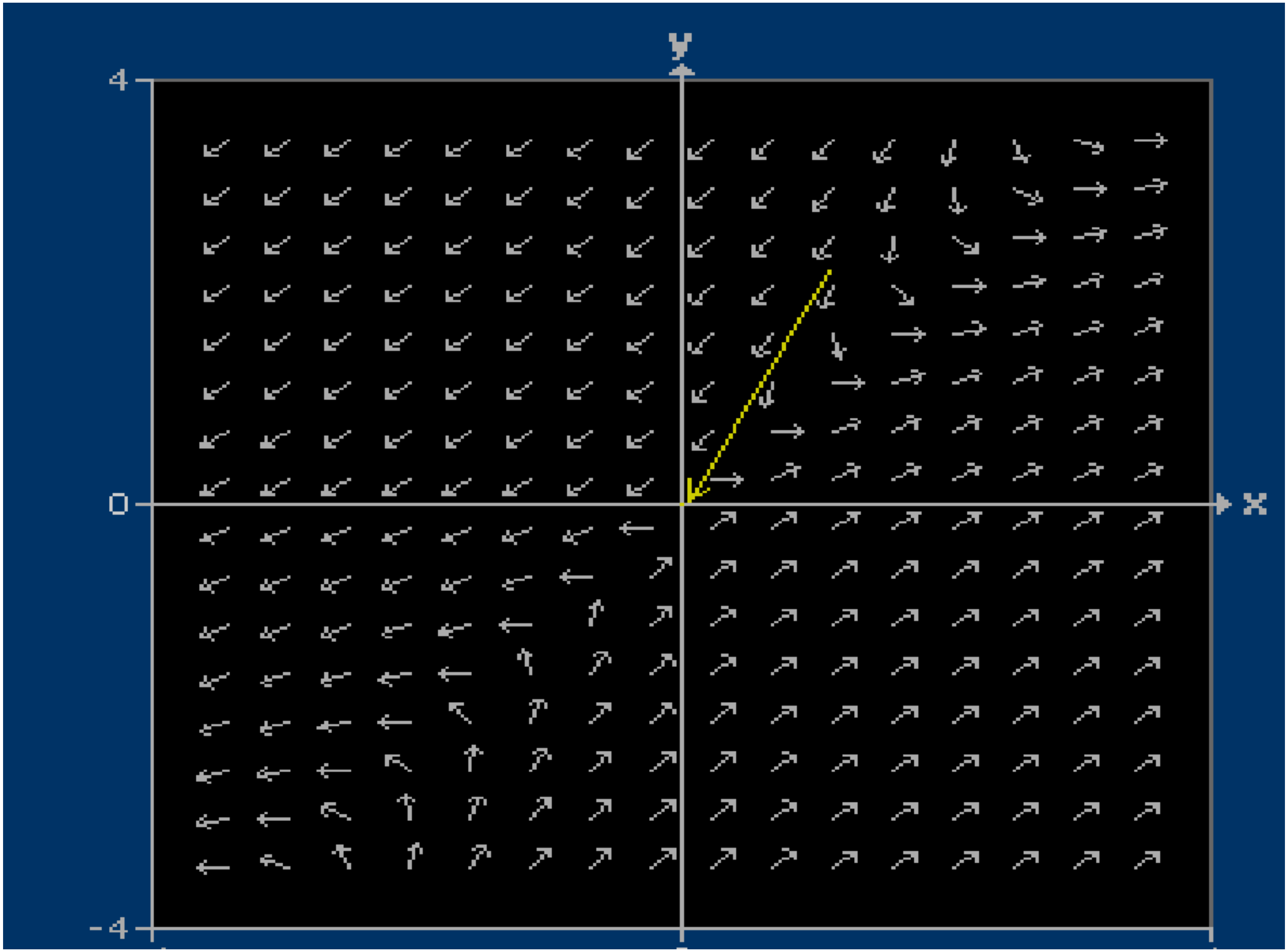

Differential Equations by Blanchard, Devaney, and Hall
HPG System Solver
Help
Clear
x = 1.00
y = -1.00
dx/dt = 17.14
dy/dt = 13.71
Overlay Time Graphs
Draw Solution
Draw Vector
dx/dt = 3*x-2*y
dy/dt = 2*x-2*y

With some adjustments for the values of t


Differential Equations by Blanchard, Devaney, and Hall
HPG System Solver
Help
Quit
y
4
0
-4
-4
0
4
x
x
4
0
-4
0
25
t
Clear
y
4
-4
0
25
t
Overlay Time Graphs
Draw Solution
Draw Vector
dx/dt = 3*x-2*y
dy/dt = 2*x-2*y
min x -4
max x 4
x0 0
min y -4
max y 4
y0 0
min t 0
max t 25
t0 0
Clear
Hide Field
Reset
Zoom Out
Zoom In
Solution
Manual delta t


Differential Equations by Blanchard, Devaney, and Hall
HPG System Solver
Help
Quit
x = 1.14
y = 0.40
dx/dt = 2.63
dy/dt = 1.49
Clear
Overlay Time Graphs
Draw Solution
Draw Vector
dx/dt = 3*x-2*y
dy/dt = 2*x-2*y
min x -4
max x 4
x0 0
min y -4
max y 4
y0 0
min t 0
max t 25
t0 0
Clear
Hide Field
Reset
Zoom Out
Zoom In
Solution
Manual delta t


y
4
0
-4
x



y
x
4
0
-4

Now let us look at a procedure that will help us obtain the solutions shown in the above graphs

A brief survey of some basic properties of matrices

Definition:

Let A be a square matrix with real number entries

$\boxed{|A| \text{ is called the determinant of } A}$

A 2x2 determinant is $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ **and is evaluated by using the equation**

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \mathbf{ad} - \mathbf{bc}$$

Example 1:

Evaluate

$$\begin{vmatrix} 2 & 1 \\ 3 & -5 \end{vmatrix} = \mathbf{2} \times (-5) - \mathbf{1} \times \mathbf{3} = \mathbf{-13}$$

A 3×3 determinant can be written in terms of 2×2 determinants in the following manner

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= \mathbf{a}_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{a}_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{a}_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= \mathbf{a}_1(b_2c_3 - b_3c_2) - \mathbf{a}_2(b_1c_3 - b_3c_1) + \mathbf{a}_3(b_1c_2 - b_2c_1)$$

Example :

Evaluate $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & -6 \\ 1 & 1 & 5 \end{vmatrix}$

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & -6 \\ 1 & 1 & 5 \end{vmatrix}$$
$$= \mathbf{2}(1 \times 5 - (-6) \times 1) - (-1)(3 \times 5 - (-6) \times 1) + \mathbf{3}(3 \times 1 - 1 \times 1)$$
$$= \mathbf{2}(5 + 6) + (15 + 6) + \mathbf{3}(3 - 1)$$
$$= \mathbf{2}(11) + \mathbf{21} + \mathbf{6}$$
$$= \mathbf{22} + \mathbf{21} + \mathbf{6}$$
$$= \mathbf{49}$$

We can extend the procedure to matrices of any size.

Matrix Multiplcation:

For this particular section, we shall only work with 2x2 matrices (a matrix with 2 rows and 2 columns) and 2x1 matrices (a matrix with 2 rows and 1 column.)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

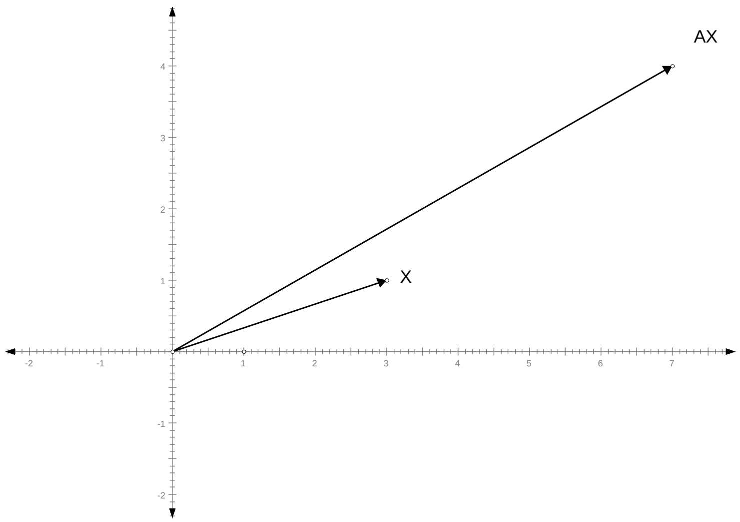
Example:

$$\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix}=\begin{bmatrix} 3\times 4+2\times 6 \\ 7\times 4+5\times 6 \end{bmatrix}=\begin{bmatrix} 24 \\ 40 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\mathbf{AX} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}=\begin{bmatrix} 7 \\ 4 \end{bmatrix}$$



a real number λ is called an eigen value of A, if we can find a non zero vector X such that $AX = \lambda X$ X is called an eigen vector for the e

Example:

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **shows that** 2 **is an eigenvalue and**

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **is an eigenvector for** 2

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

eigen value is -1, corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

....

To find the eigen values of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

just solve the equation

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \mathbf{0}$$

and follow the computation procedure shown below (very nicely done in the text book on the pages 262-268

Example of computation of Eigen Values and Eigen Vectors:

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

To find the eigenvalues (denoted by λ) , solve the equation

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = \mathbf{0}$$

$(3-\lambda)(-2-\lambda)-\mathbf{2}(-2)= \mathbf{0}$
$-(3-\lambda)(2+\lambda)+\mathbf{4} = \mathbf{0}$
$(\lambda-3)(\lambda+2)+4 = 0$
$\boldsymbol{\lambda}^2-\boldsymbol{\lambda}-\mathbf{6}+\mathbf{4} = \mathbf{0}$
$\boldsymbol{\lambda}^2-\boldsymbol{\lambda}-\mathbf{2} = \mathbf{0}$
$(\lambda-2)(\lambda+1)= \mathbf{0}$
$\boldsymbol{\lambda} = \mathbf{2}$
$\boldsymbol{\lambda} = -\mathbf{1}$

The eigenvalues are -1 **and** 2

To find the eigenvectors corresponding to -1

we have to solve the equation

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} x \\ y \end{bmatrix}$$ **for** x **and** y

that is

$$\begin{bmatrix} 3x-2y \\ 2x-2y \end{bmatrix}=\begin{bmatrix} -x \\ -y \end{bmatrix}$$

or

3x – 2y = –x → 4x – 2y = 0 → 2x – y = 0
2x – 2y = –y → 2x – y = 0

The above two equations are identical, which shows that any set of the values of x **and** y

such that $2x - y = 0$ **or** $y = 2x$ **will form the eigenvector** $\begin{bmatrix} x \\ y \end{bmatrix}$ **corresponding to** $\lambda = -1$

For example, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **is an eigenvector corresponding to** $\lambda = -1$

Because

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\begin{bmatrix} 4 \\ 2 \times 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ **is an eigenvector corresponding to** $\lambda = -1$

Because

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \end{bmatrix} = -\mathbf{1}\begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Similarly, we can find eigenvectors corresponding to the eigenvalue $\lambda = 2$

We would like to find x and y such that

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{2}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 3x - 2y \\ 2x - 2y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

3x – 2y = 2x → x – 2y = 0
2x – 2y = 2y → 2x – 4y = 0 → x – 2y = 0

therefore any set of the values of x and y

such that $x - 2y = 0$ or $x = 2y$ will form the eigenvector $\begin{bmatrix} x \\ y \end{bmatrix}$ **corresponding to $\lambda = 2$**

For example, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **is an eigenvector corresponding to $\lambda = 2$**

Because

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 10 \\ 5 \end{bmatrix}$ **is an eigenvector corresponding to $\lambda = 2$**

Because

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix} = \mathbf{2}\begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Geometrical Note:

The following graph shows the relation between v **and** Av **if** v **is an eigenvector of** A

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\mathbf{AX} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

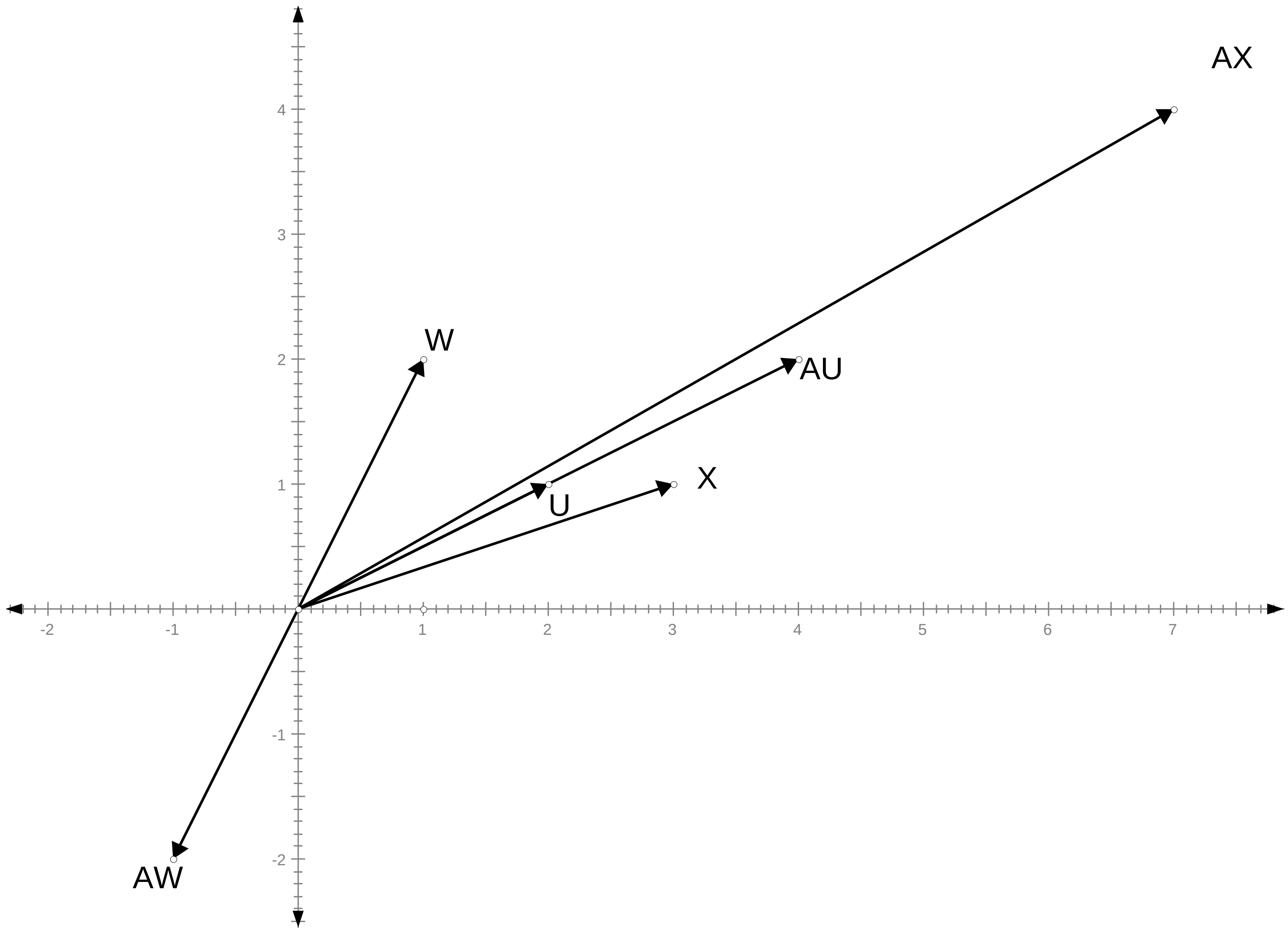
$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \mathbf{2}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{2}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

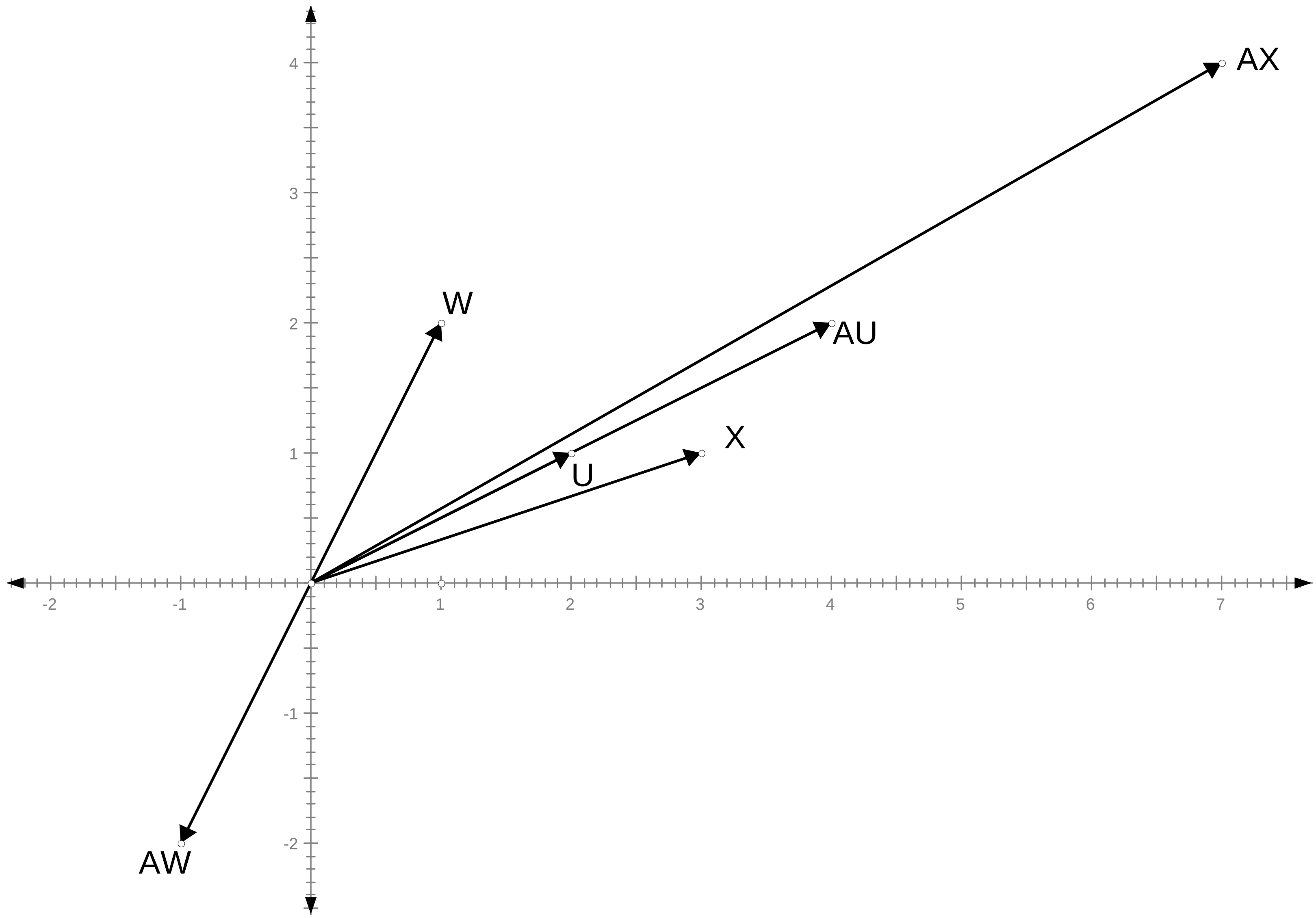
$$\mathbf{U} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{AU} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{AW} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



AX
W
AU
X
U
AW
4
3
2
1
-1
-2
-2
-1
1
2
3
4
5
6
7



Using the eigenvalues and eigen vector for solving a differential equation of the type

$\frac{dY}{dt} = AY$, **where** $Y = \begin{bmatrix} x \\ y \end{bmatrix}$ **and** A **is a 2x2 matrix with real entries**

If v **is an eigenvector of** A **corresponding to an eigen value** λ
then $ve^{\lambda t}$ **is a solution of** $\frac{dY}{dt} = AY$

Consider

$$\frac{dY}{dt} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{Y}$$

We saw that the eigenvalues of $\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$ **are** -1 **and** 2

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **is an eigenvector corresponding to** -1

therefore $\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{(-1)t} = \begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix}$ **is a solution of** $\frac{dY}{dt} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} Y$

We can check:

For $Y = \begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix}$, $\frac{dY}{dt} = \begin{bmatrix} -e^{-t} \\ -2e^{-t} \end{bmatrix}$

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ -2e^{-t} \end{bmatrix}$$

That is $\frac{dY}{dt} = AY$ **is true for** $Y = \begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix}$

Also, since $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **is an eigenvector corresponding to the eigenvalue** 2

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$ **should be a solution of** $\frac{dY}{dt} = AY$

For $Y = \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$

$$\frac{dY}{dt} = \begin{bmatrix} 4e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 4e^{2t} \\ 2e^{2t} \end{bmatrix}$$

this verifies that $\begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$ **is also a solution of** $\frac{dY}{dt} = AY$

In fact if we have two distinct eigen values of A, then any linear combination of the solutions of $\frac{dY}{dt} = AY$ corresponding to these eigen values will again be a solution of $\frac{dY}{dt} = AY$

In the example of

$$\frac{dx}{dt} = \mathbf{3x - 2y}$$
$$\frac{dy}{dt} = \mathbf{2x - 2y}$$

A general solution will look like

$$\mathbf{Y} = \begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{\alpha} \begin{bmatrix} -e^{-t} \\ -2e^{-t} \end{bmatrix} + \boldsymbol{\beta} \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$$

where α, β are real numbers.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\alpha e^{-t} + 2\beta e^{2t} \\ -2\alpha e^{-t} + \beta e^{2t} \end{bmatrix}$$

The above turns out to be a long discussion. Kindly let me know the spots that are not clear.

Let me summarize the whole thing through a different example

Takeup

the problem #8 on the page 271

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

a) To find eigenvalues, solve

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = \mathbf{0}$$

$$(2-\lambda)(1-\lambda)-((-1)(-1)) = \mathbf{0}$$

$$\lambda^2 - 3\lambda + 2 - 1 = 0$$

$$\lambda^2 - 3\lambda + 1 = 0$$

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2 \times 1}$$

or

$$\lambda = \frac{3 \pm \sqrt{5}}{2}$$

$\frac{3-\sqrt{5}}{2}$ **and** $\frac{3+\sqrt{5}}{2}$

b)

To find the corresponding (associated) eigen vectors

If $\begin{bmatrix} x \\ y \end{bmatrix}$ **is an eigenvector of** $\frac{3-\sqrt{5}}{2}$

we must have

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{3-\sqrt{5}}{2} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2x - y \\ -x + y \end{bmatrix} = \begin{bmatrix} \frac{3-\sqrt{5}}{2} x \\ \frac{3-\sqrt{5}}{2} y \end{bmatrix}$$

$2x - y = \frac{3-\sqrt{5}}{2}x \rightarrow 4x - 2y = 3x - \sqrt{5}x \rightarrow \left(1 + \sqrt{5}\right)x - 2y = 0$

$-x + y = \frac{3-\sqrt{5}}{2}y \rightarrow -2x + 2y = 3y - \sqrt{5}y \rightarrow -2x + \left(-1 + \sqrt{5}\right)y = 0$

$\left(1 + \sqrt{5}\right)x - 2y = 0$

$-2x + \left(-1 + \sqrt{5}\right)y = 0$

$\left(1 + \sqrt{5}\right)x - 2y = 0 \rightarrow y = \frac{1+\sqrt{5}}{2}x$

substitute in

$-2x + \left(-1 + \sqrt{5}\right)y = 0$

$-2x + \left(-1 + \sqrt{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)x = 0$

$0 = 0$ **what is wrong with that**

$-2x + \left(-1 + \sqrt{5}\right)y = 0 \rightarrow y = \frac{2}{-1+\sqrt{5}}x$

$\frac{1+\sqrt{5}}{2} = \frac{2}{-1+\sqrt{5}}$ **is true**

$y = \frac{1+\sqrt{5}}{2}x$

an eigen vector $\begin{bmatrix} x \\ y \end{bmatrix}$ **will look like** $\begin{bmatrix} x \\ \frac{1+\sqrt{5}}{2}x \end{bmatrix} = x\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$

$\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$ **is an associated eigen vector to the eigen value** $\frac{3-\sqrt{5}}{2}$

check:

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{1}{2}\sqrt{5} \\ -\frac{1}{2} + \frac{1}{2}\sqrt{5} \end{bmatrix}$$

$$\frac{3-\sqrt{5}}{2}\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3-\sqrt{5}}{2} \\ \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{1}{2}\sqrt{5} \\ -\frac{1}{2} + \frac{1}{2}\sqrt{5} \end{bmatrix}$$

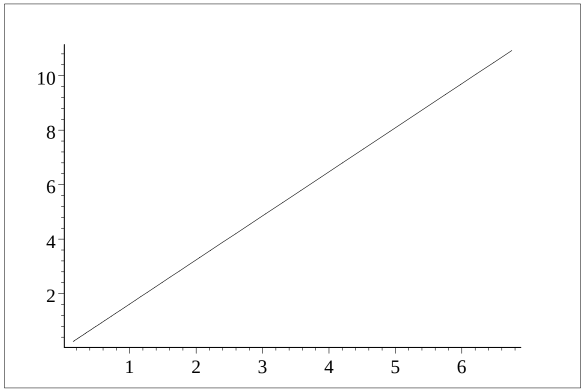
because

$$\left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$$

$\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$ **is an eigenvector corresponding to** $\frac{3-\sqrt{5}}{2}$

means that $e^{\left(\frac{3-\sqrt{5}}{2}\right)t}\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$ **is a solution**

$$\begin{bmatrix} e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \\ \frac{1+\sqrt{5}}{2} e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \end{bmatrix}$$



x $= \mathbf{e}^{\left(\frac{3-\sqrt{5}}{2}\right)t}$

y $= \frac{1+\sqrt{5}}{2}\mathbf{e}^{\left(\frac{3-\sqrt{5}}{2}\right)t}$

y $= \frac{1+\sqrt{5}}{2}\mathbf{x}$

along a striaght line with slope $\frac{1+\sqrt{5}}{2} = 1.618033989$

Take a 15 minute working session:

find another solution:

that is first find an eigen vector for the eigen value $\frac{3+\sqrt{5}}{2}$
and then write a solution
Steven

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{1}{2}\sqrt{5} \\ -\frac{1}{2} - \frac{1}{2}\sqrt{5} \end{bmatrix}$$

$$\frac{3+\sqrt{5}}{2}\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}+\frac{1}{2}\sqrt{5} \\ -\frac{1}{2}-\frac{1}{2}\sqrt{5} \end{bmatrix}$$

another solution:

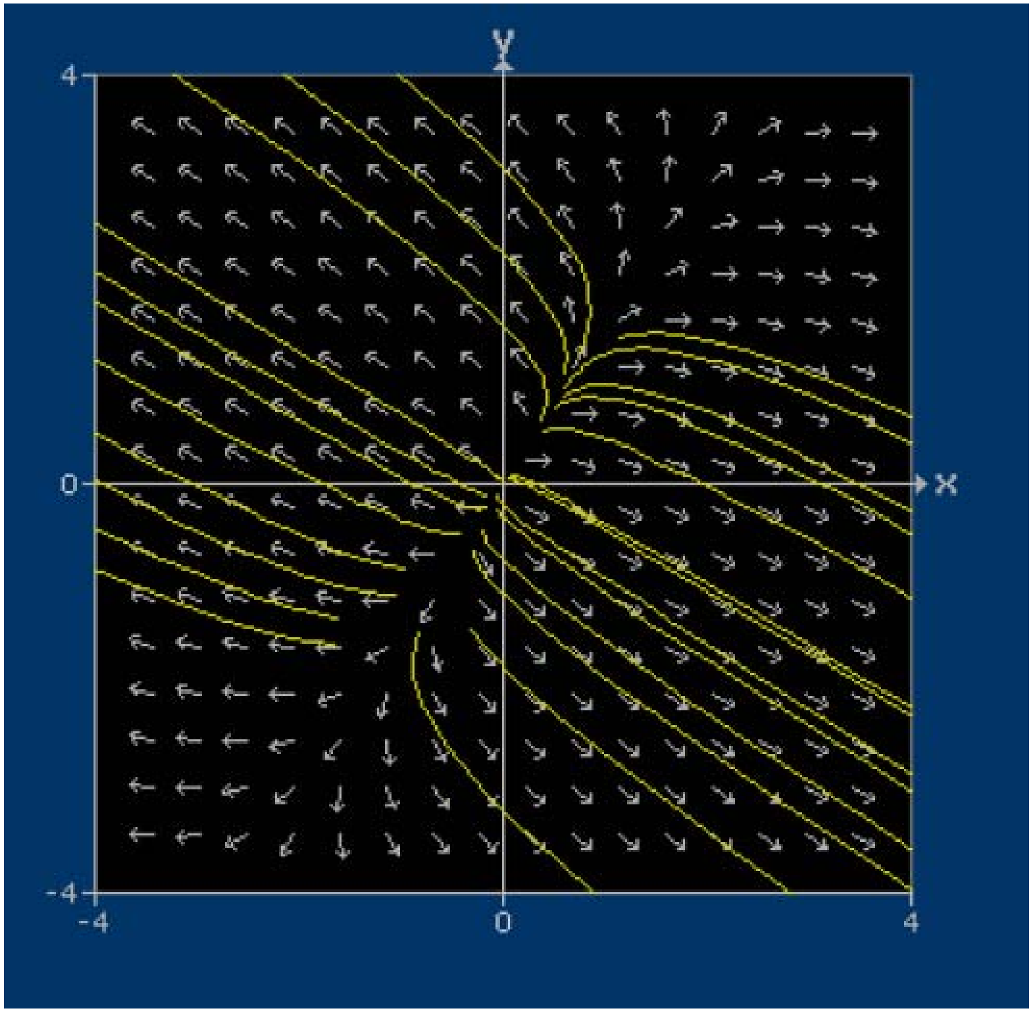
$$\mathbf{e}^{\left(\frac{3+\sqrt{5}}{2}\right)t}\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

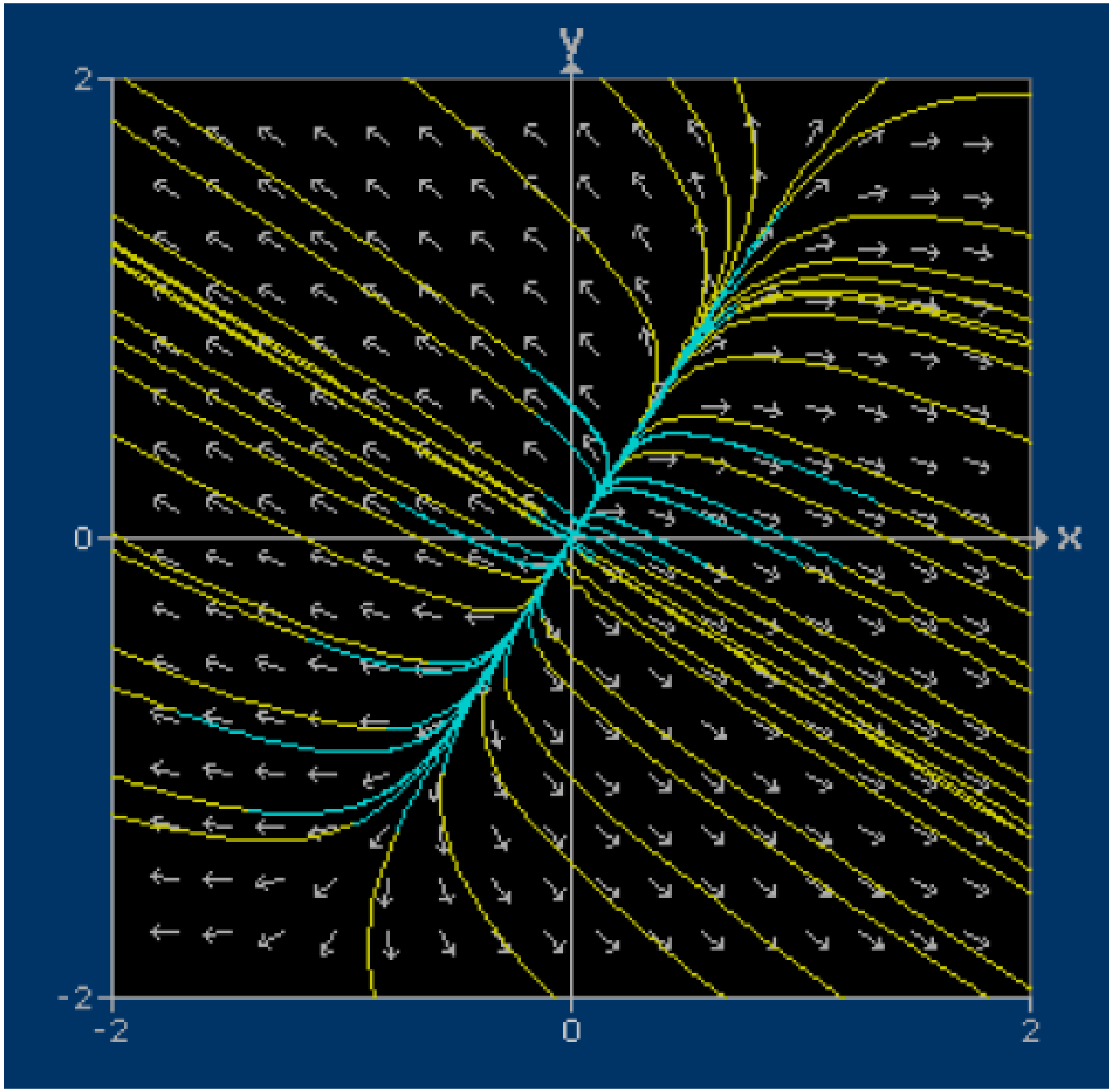
or

$$\mathbf{x} = \mathbf{e}^{\left(\frac{3+\sqrt{5}}{2}\right)t}$$

$$\mathbf{y} = \frac{1-\sqrt{5}}{2}\mathbf{e}^{\left(\frac{3+\sqrt{5}}{2}\right)t}$$

st line with slope $\frac{1-\sqrt{5}}{2} = -0.618033989$


y
x
4
0
-4
-4
0
4



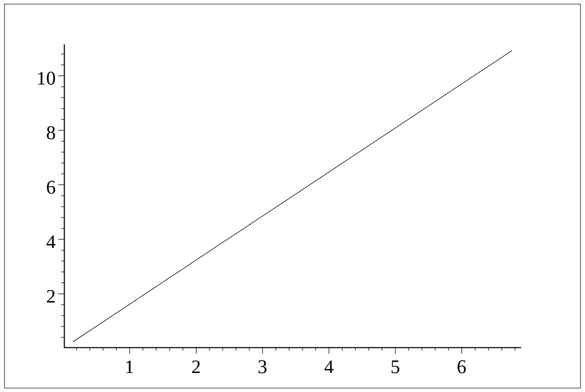
d)

For the eigenvalue $\frac{3-\sqrt{5}}{2}$ **, we have**

$\mathbf{x} = \mathbf{e}^{\left(\frac{3-\sqrt{5}}{2}\right)t}$

$\mathbf{y} = \frac{1+\sqrt{5}}{2}\mathbf{e}^{\left(\frac{3-\sqrt{5}}{2}\right)t}$

$$\begin{bmatrix} e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \\ \frac{1+\sqrt{5}}{2}e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \end{bmatrix}$$



Note that is $Y = U$ **is a solution of** $\frac{dY}{dt} = AY$ **i.e.** $\frac{dU}{dt} = AU$

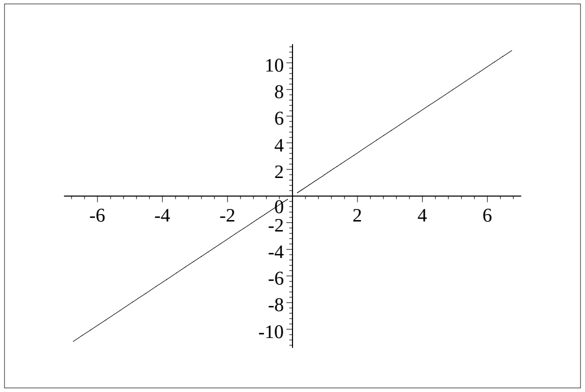
then

$Y = -U$ **is also a solution.**

because

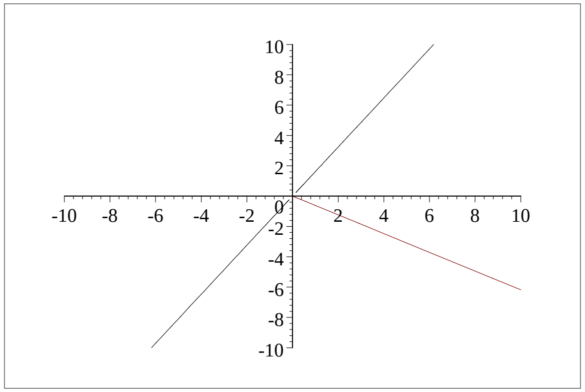
$\frac{d(-U)}{dt} = \mathbf{A}(-U)$

$$\begin{bmatrix} -e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \\ -\frac{1+\sqrt{5}}{2}e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \end{bmatrix}$$ **is also a solution**



the other one

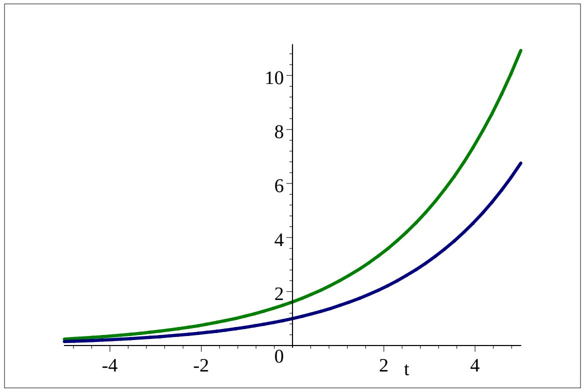
$$\begin{bmatrix} e^{\left(\frac{3+\sqrt{5}}{2}\right)t} \\ \frac{1-\sqrt{5}}{2} e^{\left(\frac{3+\sqrt{5}}{2}\right)t} \end{bmatrix}$$ **RED**



For $\left(\frac{3-\sqrt{5}}{2}\right)$

$x = e^{\left(\frac{3-\sqrt{5}}{2}\right)t}$ **Blue**

$y = \frac{1+\sqrt{5}}{2} e^{\left(\frac{3-\sqrt{5}}{2}\right)t}$ **Green**



For the eigenvalue $\left(\frac{3-\sqrt{5}}{2}\right)$

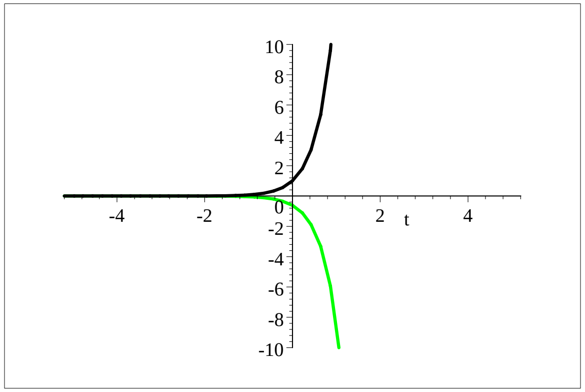
we have

$$\begin{bmatrix} e^{\left(\frac{3+\sqrt{5}}{2}\right)t} \\ \frac{1-\sqrt{5}}{2} e^{\left(\frac{3+\sqrt{5}}{2}\right)t} \end{bmatrix}$$

$x = e^{\left(\frac{3+\sqrt{5}}{2}\right)t}$ **blue**

$y = \frac{1-\sqrt{5}}{2} e^{\left(\frac{3+\sqrt{5}}{2}\right)t}$ **green**

$$x = e^{\left(\frac{3+\sqrt{5}}{2}\right)t}$$



e)

We had two distinct eigenvalues, therefore the corresponding solutions are linearly independent (that is they are not scalar multiples of each other in this case of only two dimensions)

$$\begin{bmatrix} e^{\left(\frac{3+\sqrt{5}}{2}\right)t} \\ \frac{1-\sqrt{5}}{2} e^{\left(\frac{3+\sqrt{5}}{2}\right)t} \end{bmatrix} \textbf{ and } \begin{bmatrix} e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \\ \frac{1+\sqrt{5}}{2} e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \end{bmatrix}$$

any linear combination of these two solutions is also a solution

that is, the general solution may be written as

$$\mathbf{Y} = \boldsymbol{\alpha} \begin{bmatrix} e^{\left(\frac{3+\sqrt{5}}{2}\right)t} \\ \frac{1-\sqrt{5}}{2} e^{\left(\frac{3+\sqrt{5}}{2}\right)t} \end{bmatrix} + \boldsymbol{\beta} \begin{bmatrix} e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \\ \frac{1+\sqrt{5}}{2} e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \end{bmatrix}$$

Example:

$$\mathbf{2}\begin{bmatrix} e^{\left(\frac{3+\sqrt{5}}{2}\right)t} \\ \frac{1-\sqrt{5}}{2}e^{\left(\frac{3+\sqrt{5}}{2}\right)t} \end{bmatrix} - \mathbf{3}\begin{bmatrix} e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \\ \frac{1+\sqrt{5}}{2}e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{\left(\frac{3-\sqrt{5}}{2}\right)t} - 3e^{\left(\frac{3-\sqrt{5}}{2}\right)t} \\ \left(1-\sqrt{5}\right)e^{\left(\frac{3+\sqrt{5}}{2}\right)t} - 3\left(\frac{1+\sqrt{5}}{2}e^{\left(\frac{3-\sqrt{5}}{2}\right)t}\right) \end{bmatrix}$$

Now consider:

14 on the page 272

Given that

$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$**, eigenvalues:** $2, 3$

$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$**, eigenvectors:** $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} \leftrightarrow 3, \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 2$

Solutions:

$\mathbf{e}^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{3t} \\ e^{3t} \end{bmatrix}$

and

$\mathbf{e}^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$

general solution is $k_1\begin{bmatrix} 2e^{3t} \\ e^{3t} \end{bmatrix} + k_2\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$

a)

$t = 0$**:** $x = 1, y = 0$

$$\mathbf{k}_1\begin{bmatrix} 2e^{3(0)} \\ e^{3(0)} \end{bmatrix}+\mathbf{k}_2\begin{bmatrix} e^{2(0)} \\ e^{2(0)} \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{k}_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}+\mathbf{k}_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2k_1 + k_2 \\ k_1 + k_2 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

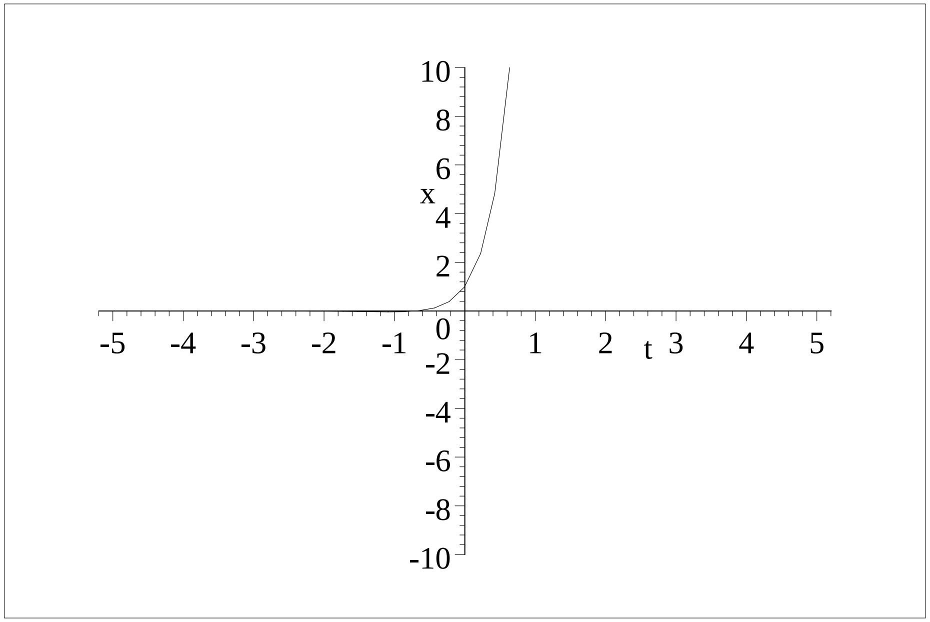
$\mathbf{2k}_1+\mathbf{k}_2= \mathbf{1}$

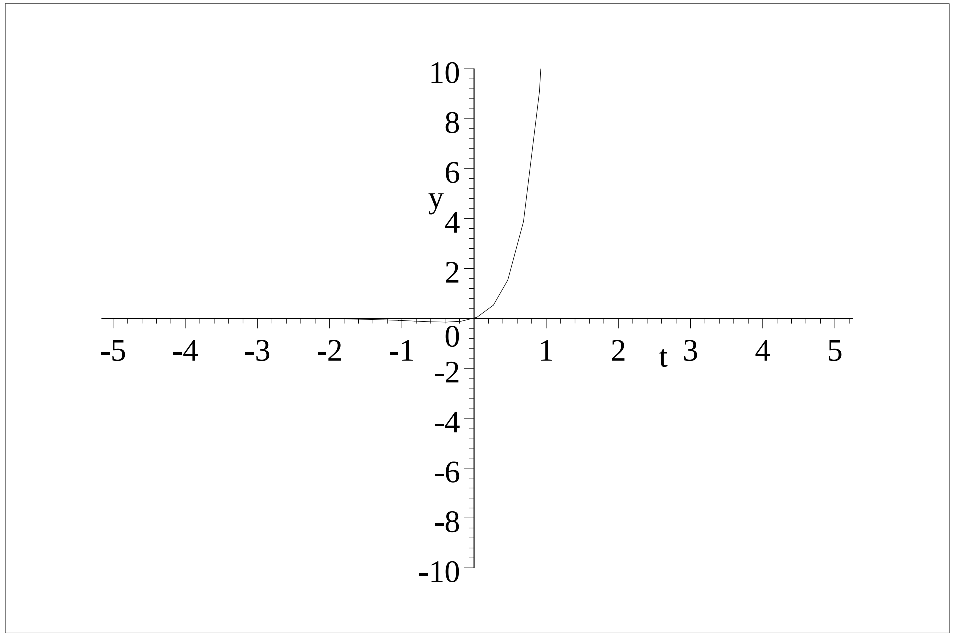
$\mathbf{k}_1+\mathbf{k}_2= \mathbf{0}$

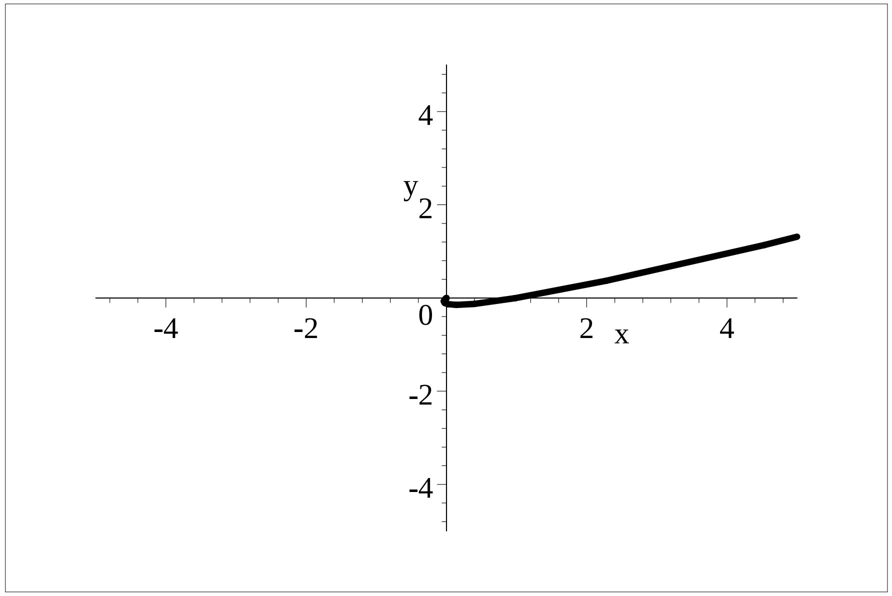
$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$k_1 = 1$**,** $k_2 = -1$

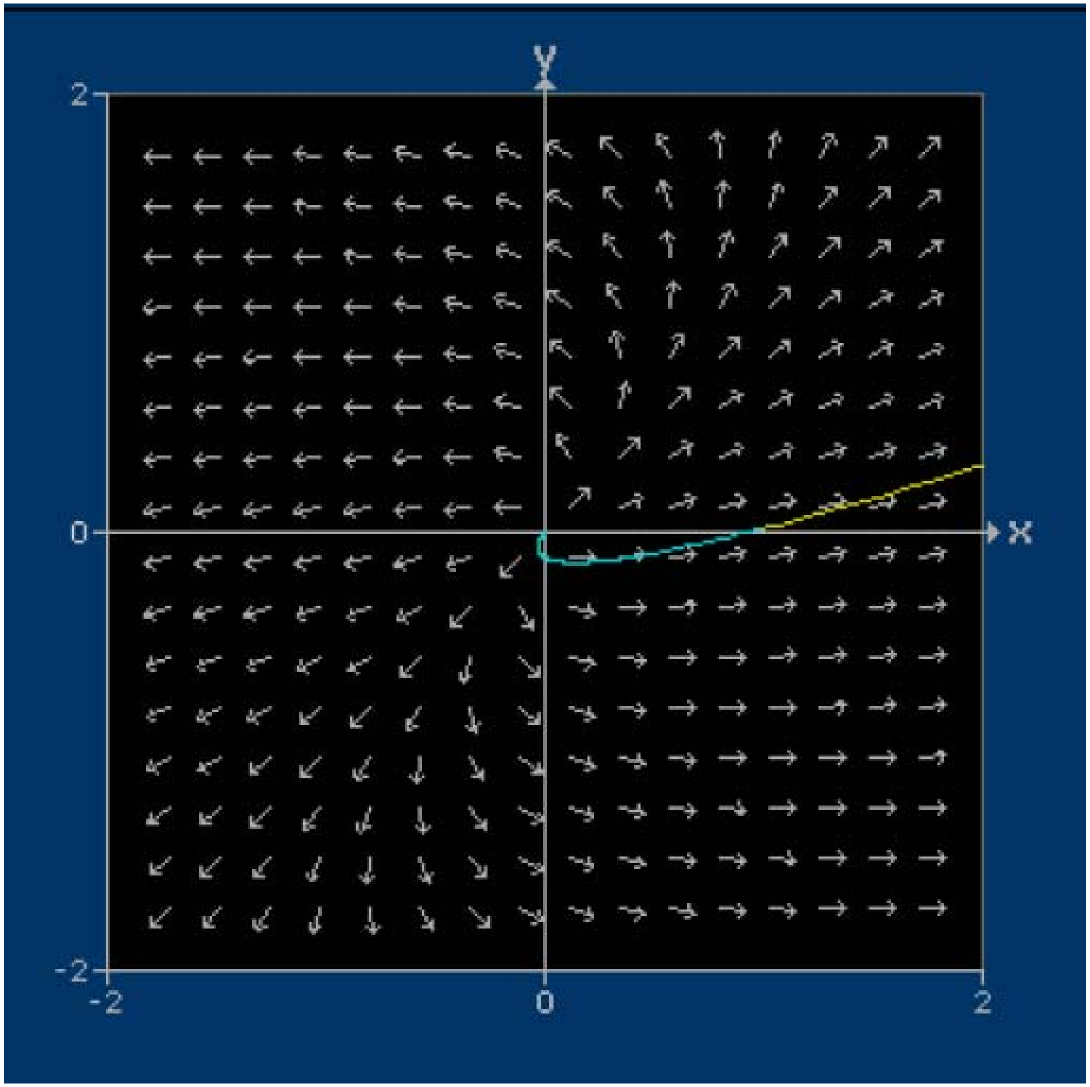
$$\mathbf{Y} = \begin{bmatrix} 2e^{3t} \\ e^{3t} \end{bmatrix}-\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}=\begin{bmatrix} 2e^{3t} - e^{2t} \\ e^{3t} - e^{2t} \end{bmatrix}$$


10
8
6
x
4
2
0
-2
-4
-6
-8
-10
-5
-4
-3
-2
-1
1
2
t
3
4
5


10
8
6
y
4
2
0
-2
-4
-6
-8
-10
-5
-4
-3
-2
-1
1
2
t
3
4
5



b) and c) do yourself please



Suggested Practice Problems:

1 thru 19, odd numbered on the pages 271-273