Section 5.1, please read and try the suggested practice problems

Natural Logarithm: Recall:

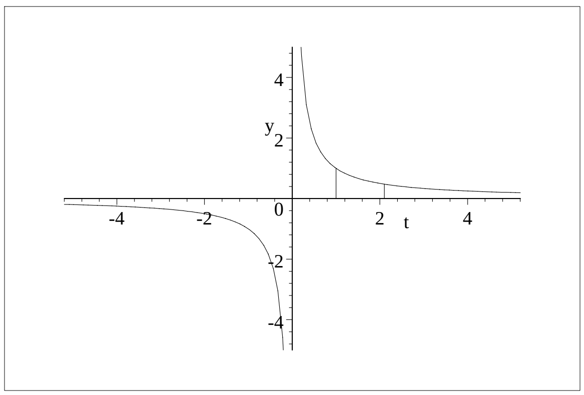
$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \;\; n \neq -1$$

$$\int \frac{1}{x} dx = \int x^{-1} dx = \frac{x^{-1+1}}{-1+1} + C$$ **undefined**

What is $\int \frac{1}{x} dx$ **?**

A function whose derivative is $\frac{1}{x}$

—



$$\mathbf{F}(\mathbf{x}) = \int_1^x \frac{1}{t}\,\mathbf{dt}$$

by
theSecond FundamentalTheorem of Calculus
(Recall from the chapter 4)

$$\mathbf{F}'(\mathbf{x}) = \frac{1}{x}$$

$$\int \frac{1}{x}\mathbf{dx} = \mathbf{F}(\mathbf{x}) + \mathbf{C}$$

$$\int \frac{1}{x}\mathbf{dx} = \int_1^x \frac{1}{t}\mathbf{dt} + \mathbf{C}$$

because of the continuity of of f(t)=$\frac{1}{t}$
on $(0,\infty)$**, we have**

$\int_1^x \frac{1}{t}dt$ **as a well defined function.**

Section 5.1, please read and try the suggested practice problems

$\ln(x)=\int_1^x \frac{1}{t}\mathbf{dt}$

$\int_1^x \frac{1}{t}\mathbf{dt}$

as the natural log of x.

$\ln(x)=\int_1^x \frac{1}{t}\mathbf{dt}$

$\ln(0)$ **undefined**

$\ln(1)=\int_1^1 \frac{1}{t}\mathbf{dt} = \mathbf{0}$

$\ln(-2)$

$\int_1^{-2} \frac{1}{t}dt = -\int_{-2}^1 \frac{1}{t}dt$, **improper and also it does not exist...**

$\ln(x) = \int_1^x \frac{1}{t}dt$ **,** $x > 0$

$\int \frac{1}{x}\mathbf{dx} = \ln \mathbf{x} + \mathbf{C}$

$\frac{d(\ln x)}{dx}=\frac{1}{x}+\mathbf{C}$

....

Shall fill the following:

$\frac{d(\ln(|x|) + C)}{dx}=\frac{1}{x}$

$\int \frac{1}{x}\mathbf{dx} = \ln|x|+\mathbf{C}$

shall post the proof later

Example 1

Section 5.1, please read and try the suggested practice problems

To find

$\frac{dy}{dx}$ **if** $y = x^2 \ln x$

$$\mathbf{y} = \mathbf{x}^2 \ln \mathbf{x}$$
$$\frac{dy}{dx} = (2x)\ln \mathbf{x} + \mathbf{x}^2\left(\frac{1}{x}\right)$$
$$\frac{dy}{dx} = \mathbf{2x}\ln \mathbf{x} + \mathbf{x}$$

Example 2:

Find $f'\left(\frac{\pi}{4}\right)$ **if** $f(x) = \ln \cos x$

remember that

$$\mathbf{y} = \ln \mathbf{u}$$
$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$\mathbf{f}'(\mathbf{x}) = \frac{1}{\cos x}(-\sin x)$$
$$\mathbf{f}'(x) = -\frac{\sin x}{\cos x}$$
$$\mathbf{f}'(x) = -\tan \mathbf{x}$$
$$\mathbf{f}'\left(\frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -\mathbf{1}$$

....

extension:

Find an equation of the tangent line at $\left(\frac{\pi}{4}, -\frac{1}{2}\ln 2\right)$

recall:

$$\frac{y - \left(-\frac{1}{2}\ln 2\right)}{x - \frac{\pi}{4}} = -\mathbf{1}$$

$\mathbf{y} - \left(-\frac{1}{2}\ln 2\right) = -\mathbf{1}\left(x - \frac{\pi}{4}\right)$

Section 5.1, please read and try the suggested practice problems

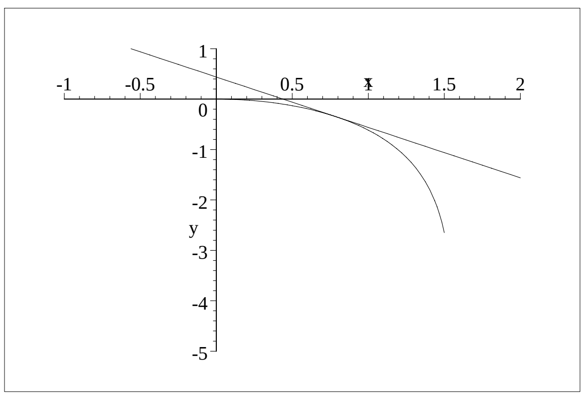
OR

$\mathbf{y} = (-1)\mathbf{x} + \mathbf{b}$
$\mathbf{y} = -\mathbf{x} + \mathbf{b}$

$\left(\frac{\pi}{4}, -\frac{1}{2}\ln 2\right)$ **is on the line**
$-\frac{1}{2}\ln 2 = -\frac{\pi}{4} + b$
$\mathbf{b} = \frac{\pi}{4} - \frac{1}{2}\ln \mathbf{2}$

$\mathbf{y} = -\mathbf{x} + \frac{\pi}{4} - \frac{1}{2}\ln \mathbf{2}$

to the graph of $f(x) = \ln\cos x$
$f(x) = \ln\cos x$



The Derivative of $\ln|x|$ **is** $\frac{1}{x}$ $\quad , x \neq 0$

An antiderivative of $\frac{1}{x}$ **is** $\ln|x| + C$

$\int \frac{du}{u} = \ln|u| + \mathbf{C}$

Section 5.2, please read and try the suggested practice problems

Example 3: To evaluate

$\int \frac{-1}{x(\ln x)^2}\,\mathbf{dx}$

$\ln \mathbf{x} = \mathbf{u}$
$\rightarrow \frac{1}{x}\mathbf{dx} = \mathbf{du}$

$\int \frac{-1}{x(\ln x)^2}\mathbf{dx}$
$= \int -\frac{1}{u^2}du$
$= \int -u^{-2}du$
$= -\frac{u^{-2+1}}{-2+1} + C$
$= \frac{1}{u} + C$
$= \frac{1}{\ln x} + C$

Section 5.2, please read and try the suggested practice problems

Example 4: To evaluate

$\int_0^1 \frac{t+1}{2t^2+4t+3}\mathbf{dt}$

$2t^2 + 4t + 3 = u \qquad , t = 0 : u = 3, \; t = 1 : u = 9$
$\rightarrow (4t+4)\mathbf{dt} = \mathbf{du}$
$\rightarrow \mathbf{4}(t+1)\mathbf{dt} = \mathbf{du}$
$\rightarrow (t+1)\mathbf{dt} = \frac{1}{4}\mathbf{du}$

$\int_3^9 \frac{1}{4}\frac{du}{u}$
$= \frac{1}{4}\ln u|_3^9$
$= \frac{1}{4}(\ln 9 - \ln 3)$
$= \frac{1}{4}(\ln 3^2 - \ln 3) \qquad \frac{1}{4}(\ln 3^2 - \ln 3) = \frac{1}{4}\ln\left(\frac{3^2}{3}\right)$
$= \frac{1}{4}(2\ln 3 - \ln 3)$
$= \frac{1}{4}\ln 3$

Example 5:

To find $\frac{dy}{dx}$

Section 5.2, please read and try the suggested practice problems

if $y = \dfrac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}}$

$\ln y = \ln\left(\dfrac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}}\right)$

$\rightarrow \ln \mathbf{y} = \ln\left((x^2+3)^{2/3}(3x+2)^2\right) - \ln\sqrt{x+1}$

$\rightarrow \ln \mathbf{y} = \ln\left((x^2+3)^{2/3}\right) + \ln(3x+2)^2 - \ln(x+1)^{1/2}$

$\rightarrow \ln \mathbf{y} = \frac{2}{3}\ln(x^2+3) + \mathbf{2}\ln(3x+2) - \frac{1}{2}\ln(x+1)$

differentiate with respect to x

$$\boxed{\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{3}\cdot\frac{1}{x^2+3}(2x) + 2\cdot\frac{1}{3x+2}(3) - \frac{1}{2}\cdot\frac{1}{x+1}$$

$$\frac{dy}{dx} = \left(\frac{2}{3}\cdot\frac{1}{x^2+3}(2x) + 2\cdot\frac{1}{3x+2}(3) - \frac{1}{2}\cdot\frac{1}{x+1}\right)\mathbf{y}$$

$$\frac{dy}{dx} = \left(\frac{2}{3}\cdot\frac{1}{x^2+3}(2x) + 2\cdot\frac{1}{3x+2}(3) - \frac{1}{2}\cdot\frac{1}{x+1}\right)\left(\frac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}}\right)$$

More Examples:

Example6:

$$\int \frac{(\ln x)^3}{x}dx$$

$\ln x = u \rightarrow \frac{1}{x}dx = du$

$$\int \frac{(\ln x)^3}{x}dx$$

$= \int u^3 du$

$= \frac{u^4}{4} + C$

$= \frac{(\ln x)^4}{4} + C$

Example7: Let us work on

$$\int \tan x dx$$

$\int \tan x dx$
$= \int \frac{\sin x}{\cos x} dx$
$\cos x = u \rightarrow -\sin x dx = du$
$\int \tan x dx$
$= \int \frac{\sin x}{\cos x} dx$
$= -\int \frac{du}{u}$
$= -\ln|u| + C$
$= -\ln|\cos x| + C$

Section 5.2, please read and try the suggested practice problems

another way
$-\ln|\cos x| = -1 \times \ln|\cos x| = \ln|\cos x|^{-1} = \ln|\sec x|$

Example8:

To illustrate the procedure of the logarithmic differentiation

To differentiate
$$y = \frac{\sqrt[3]{(1+x^2)}\,(1+x^4)^2}{\sqrt{1+2x^2}}$$
Quotient rule will not be too pleasant

$$y = \frac{\sqrt[3]{(1+x^2)}\,(1+x^4)^2}{\sqrt{1+2x^2}}$$
$$\rightarrow \ln y = \ln \frac{\sqrt[3]{(1+x^2)}\,(1+x^4)^2}{\sqrt{1+2x^2}}$$
$$\rightarrow \ln y = \ln\left(\sqrt[3]{(1+x^2)}\,(1+x^4)^2\right) - \ln\sqrt{1+2x^2}$$
$$\rightarrow \ln y = \ln\sqrt[3]{(1+x^2)} + \ln(1+x^4)^2 - \ln\sqrt{1+2x^2}$$
$$\rightarrow \ln y = \ln(1+x^2)^{1/3} + \ln(1+x^4)^2 - \ln(1+2x^2)^{1/2}$$
$$\rightarrow \ln y = \frac{1}{3}\ln(1+x^2) + 2\ln(1+x^4) - \frac{1}{2}\ln(1+2x^2)$$

now, we may differentiate with respect to x
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3} \cdot \frac{2x}{1+x^2} + 2\frac{4x^3}{1+x^4} - \frac{1}{2}\frac{4x}{1+2x^2}$$
$$\rightarrow \frac{dy}{dx} = \left(\frac{1}{3} \cdot \frac{2x}{1+x^2} + 2\frac{4x^3}{1+x^4} - \frac{1}{2}\frac{4x}{1+2x^2}\right)y$$

$$\rightarrow \frac{dy}{dx} = \left(\frac{1}{3} \cdot \frac{2x}{1+x^2} + 2\frac{4x^3}{1+x^4} - \frac{1}{2}\frac{4x}{1+2x^2}\right)\frac{\sqrt[3]{(1+x^2)}\,(1+x^4)^2}{\sqrt{1+2x^2}}$$

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Section 5.2, please read and try the suggested practice problems

$\ln(x-2) - \ln(x+2) + 2\ln x$

$=\ln\left(\frac{x-2}{x+2}\right) + \ln x^2$

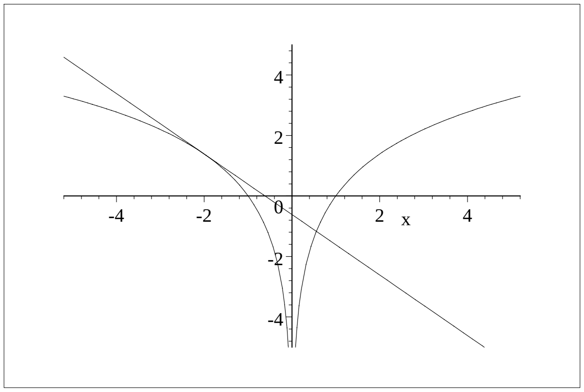
$=\ln\left(\left(\frac{x-2}{x+2}\right)x^2\right)$

$=\ln\left(\frac{x^2(x-2)}{x+2}\right)$

Example9:

Find the slope of the tangent line to the graph of $y = \ln x^2$ at $(-2, \ln 4)$

$y = \ln x^2$



$\frac{dy}{dx} = \frac{1}{x^2}(2x) \quad \rightarrow \quad \frac{dy}{dx} = \frac{2}{x}$

Slope of the tangent line at $(-2, \ln 4)$ is $\frac{2}{-2} = -1$

equation is $y = -x + b$

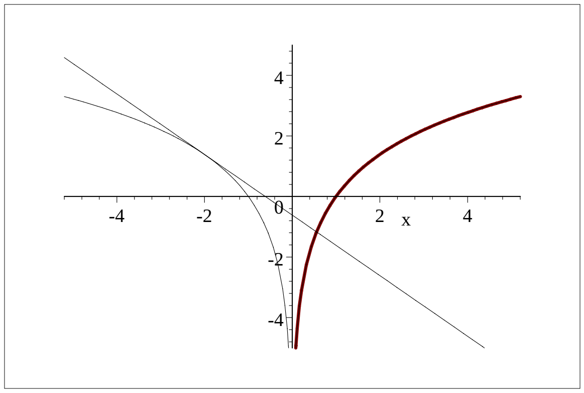
$\ln 4 = 2 + b$

$b = \ln 4 - 2$

$y = -x + \ln 4 - 2$

....

$y = \ln x^2 \rightarrow y = 2\ln x$



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Example 10:

Section 5.2, please read and try the suggested practice problems

$$y = \ln(3x-7)^4$$
$$\frac{dy}{dx} = \frac{1}{(3x-7)^4}4(3x-7)^3 3 = \frac{12}{3x-7}$$

$y = 4\ln(3x-7)$ on the appropriate domain

$$\frac{dy}{dx} = 4\frac{3}{3x-7}$$
$$\frac{dy}{dx} = \frac{12}{3x-7}$$

....

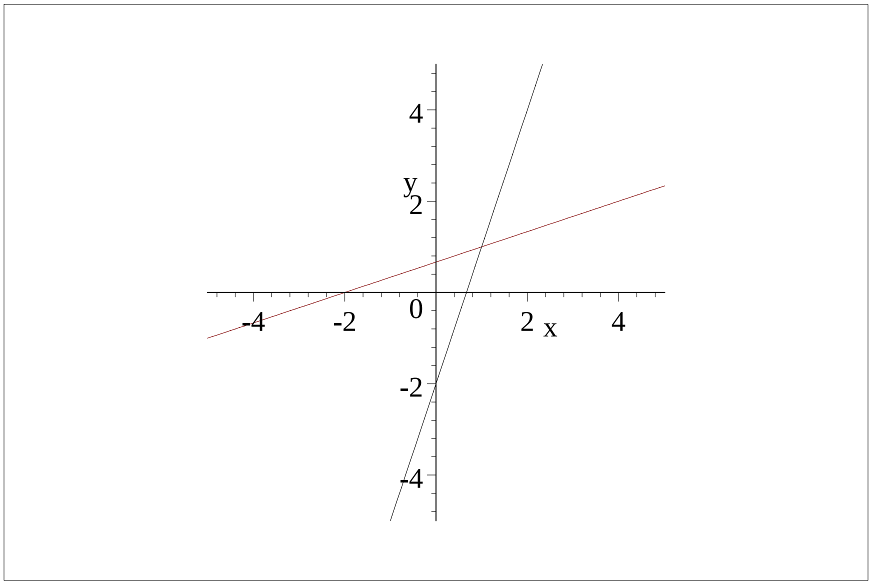
Inverse Functions

Consider the functions

$f(x) = 3x - 2$

$g(x) = \frac{x+2}{3}$

$f(x) = 3x - 2$

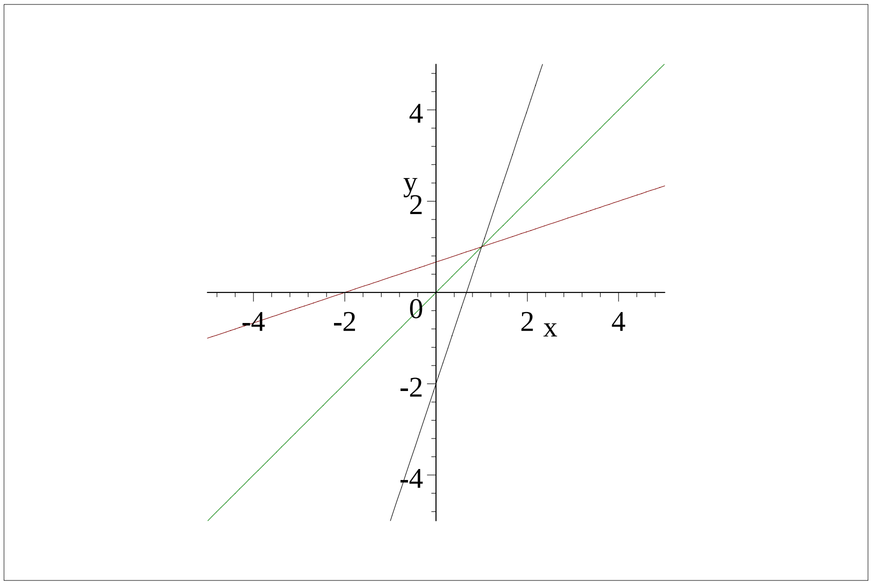


Which one is a graph of f and which one is a graph of g?

black: f, red: g

$g \circ f(x) = g(f(x)) = g(3x - 2) = \frac{(3x-2)+2}{3} = \frac{3x}{3} = x$

$f \circ g(x) = f(g(x)) = f\left(\frac{x+2}{3}\right) = 3\left(\frac{x+2}{3}\right) - 2 = x$



slope of f is 3
slope of g is $\frac{1}{3}$
The green line is a graph of $y = x$

Note that the white and the green graphs are reflections of each other by the green graph.

Definition:

Let f be a function with the domain D and the range R
and g be a function with the domain R and the range D

and

$g \circ f(x) = x$

and

$f \circ g(x) = x$

Then

f and g are called inverses of each other.

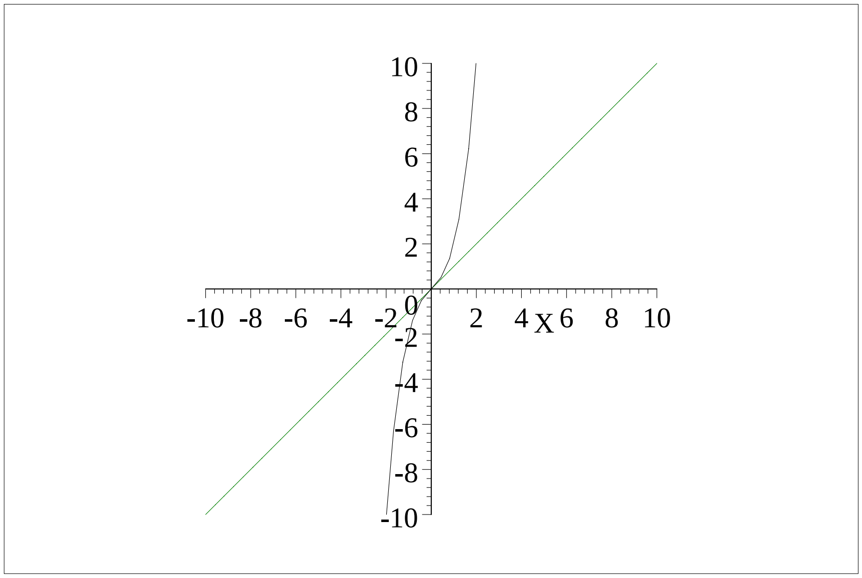
$g = f^{-111}$

$f = g^{-1}$

caution:
$f^{-1} \neq \frac{1}{f}$

To obtain a graph of the inverse of f from a graph of f,
reflect the graph of f by the line y=x

$f(x) = x^3 + x$



Saw that the slope of the tangent line at (1,2) to the graph of f is 4
The slope of the tangent line at $(2, 1)$ to the graph of f^{-1} is $\frac{1}{4}$

In general
Inverse Function Theorem
Under certain conditions (to be specified later)

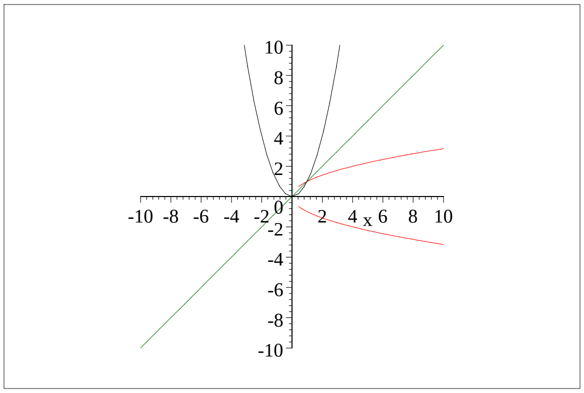
If $g = f^{-1}$
and (a, b) is on the graph of f, note that the slope of the tangent at (a,b) to the graph of f is $f'(a)$

then (b, a) is on the graph of $g = f^{-1}$
the slope of the tangent at (b,a) to the graph of $g = f^{-1}$ is $\frac{1}{f'(a)}$

$(f^{-1})'(b) = \frac{1}{f'(a)}$, provided $f'(a) \neq 0$

Example of a function that does not have an inverse:

$f(x) = x^2$



Does not have an inverse because does not pass the horizontal line test

Remember the following:
Inverse:

1. (a,b) is on f , then (b,a) is on f^{-1}
2. reflection by y=x rule
3. horizontal line test

or

4. in order for f to have an inverse the graph will have to be ↗ or ↘ that is either $f'(x) > 0$ or $f'(x) < 0$
5. finding an expression for the inverse
6. Inverse function theorem

Find the inverse of

1. $f(x) = 4x - 2$

$y = 4x - 2$

$x \leftrightarrow y$

$x = 4y - 2$

Solve for y

$x = 4y - 2$

$\mathbf{x + 2 = 4y}$

$\rightarrow \frac{x+2}{4} = \mathbf{y}$

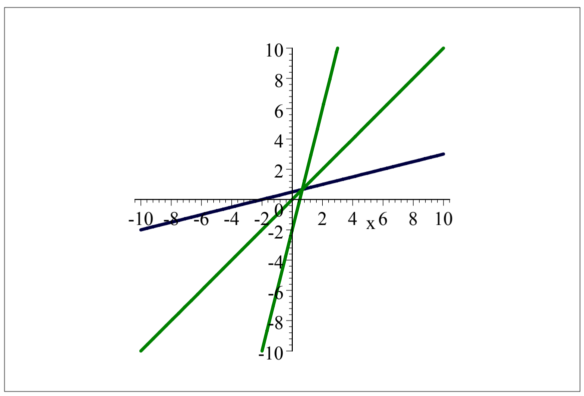
$\rightarrow \mathbf{f}^{-1}(x) = \frac{1}{4}\mathbf{x} + \frac{1}{2}$

Note that the slopes of f **and** f^{-1} **are the reciprocals of each other**

The graphs of f **and** f^{-1} **are reflections of each other by the line**

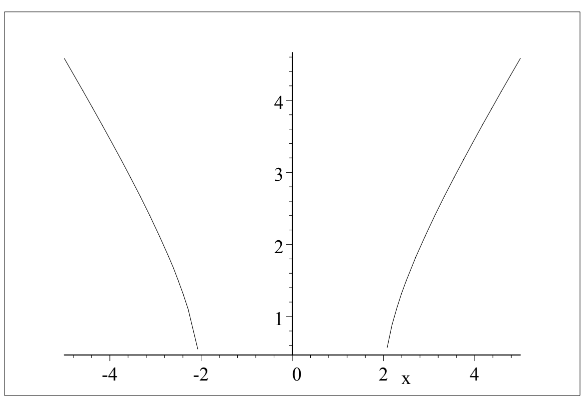
$y = x$

$f(x) = 4x - 2$



Example 2:

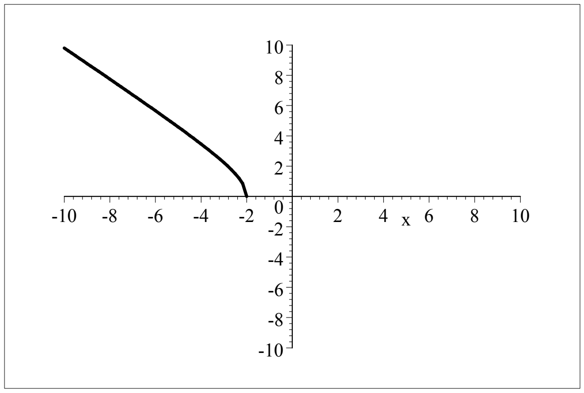
$f(x) = \sqrt{x^2 - 4}$ **is a function**



but does not have an inverse because it fails the horizontal line test

But

$f(x) = \sqrt{x^2 - 4}$**,** $x \le -2$
Domain: $(-\infty, -2]$ **,the range is** $[0, \infty)$

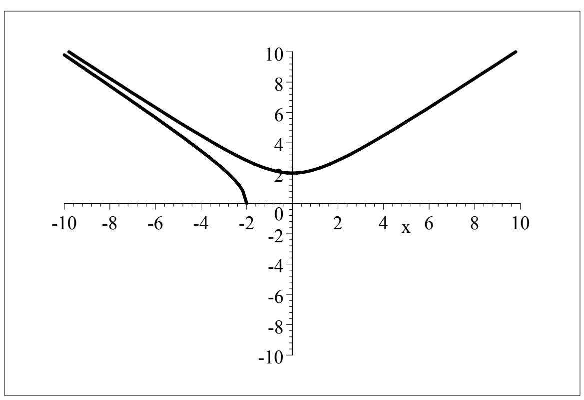


passes the horizontal line test (of course it has to pass the VLT also) therefore does have an inverse.
f^{-1} : **Domain** $[0, \infty)$ **and the range is** $(-\infty, -2$

$y = \sqrt{x^2 - 4}$ **where** $x \le -2$

x $\leftrightarrow$ **y**

$x = \sqrt{y^2 - 4}$ **where** $y \le -2$
$\mathbf{x}^2 = \mathbf{y}^2 - 4$
$\mathbf{y}^2 = \mathbf{x}^2 + 4$
y $= \sqrt{x^2 + 4}$



What went wrong?

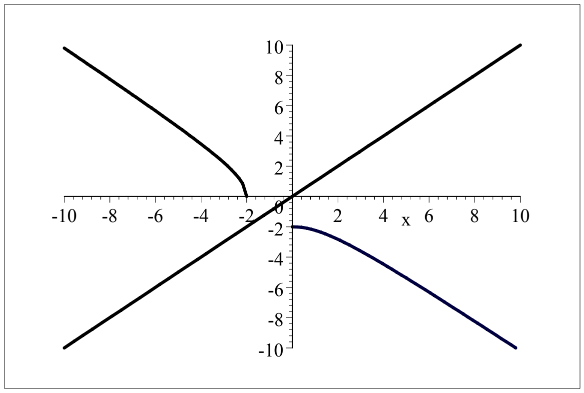
Note that y **is negative**

therefore for us,
$\mathbf{y}^2 = \mathbf{x}^2 + 4$

→

y $= -\sqrt{x^2+4}$

f$^{-1}(x) = -\sqrt{x^2+4}$



Find the slope of f at $\left(-4, 2\sqrt{3}\right)$

f(**x**) $= \sqrt{x^2-4}$

f$'$(**x**) $= \dfrac{x}{\sqrt{x^2-4}}$

the slope of f at $\left(-4, 2\sqrt{3}\right)$

is $\dfrac{(-4)}{\sqrt{(-4)^2-4}} = -\dfrac{2}{3}\sqrt{3}$

Find the slope of

$f^{-1}(x) = -\sqrt{x^2+4}$ **at** $\left(2\sqrt{3}, -4\right)$

$(f^{-1})'(x) = -\dfrac{x}{\sqrt{x^2+4}}$

the slope at $\left(2\sqrt{3}, -4\right)$

is $-\dfrac{2\sqrt{3}}{\sqrt{\left(2\sqrt{3}\right)^2+4}} = -\dfrac{1}{2}\sqrt{3} = -\dfrac{3}{2\sqrt{3}}$

This was an illustration of the Inverse Function Theorem:

If $f(a) = b$

$(f^{-1})'(b) = \dfrac{1}{f'(a)}$

provided $f'(a) \neq 0$

make sure to read the requirenents for the inverse function theorem in the notes and the text

Question 1:

Does $f(x) = x^5 + x^3 + x$ **have an inverse?**

Yes, because it is one one (graph passes the HLT)

Or recall the first derivative test:

If $f'(x) > 0$ **the function is** $\nearrow$

Here

$\mathbf{f'(x) = 5x^4+3x^2+1 > 0}$

therefore the function is 1-1 and has an inverse

Question 2:

Find the slope of f^{-1} **at** $(3, 1)$

if $f(x) = x^5 + x^3 + x$

Recall that $(f^{-1})'(3) = \dfrac{1}{f'(1)}$

Find

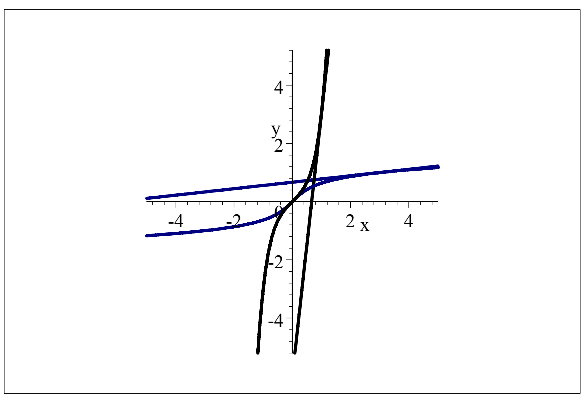
$f'(x) = 5x^4 + 3x^2 + 1$

and since

$\mathbf{f'(1) = 5(1)^4+3(1)^2+1 = 9}$

$(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{9}$

$y = x^5 + x^3 + x$



Equation of the tangent at $(1, 3)$ **for** f

$(y - 3) = \mathbf{9}(x - 1)$

Equation of the tangent at $(3, 1)$ **for** f^{-1}

$(y - 1) = \frac{1}{9}(x - 3)$

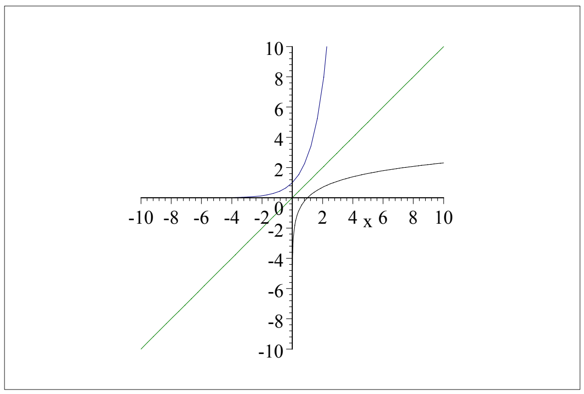
Exponential Functions

Section 5.4, please read and work on the suggested practice problems in the text

Since $f(x) = \ln x$ **is one-one ,**
f an inverse.

The Domain of $f(x) = \ln x$ **is** $(0,\infty)$
The Range of $f(x) = \ln x$ **is** $(-\infty,\infty)$

$\ln x$



We represent $f^{-1}(x) = e^x$
the domain of $y = e^x$ **is** $(-\infty,\infty)$
the range of $y = e^x$ **is** $(0,\infty)$

......
If
$y = e^x$ **then** $\ln y = \ln e^x = x$

Find $\frac{dy}{dx}$ **if** $y = e^x$
y $=$ **e**^{x}
$\rightarrow \ln$**y** $=$ **x**
differentiate with respect to x
$\frac{1}{y}\frac{dy}{dx} = 1$
$\rightarrow$
$\frac{dy}{dx} = y$
$\rightarrow$
$\frac{dy}{dx} = e^x$

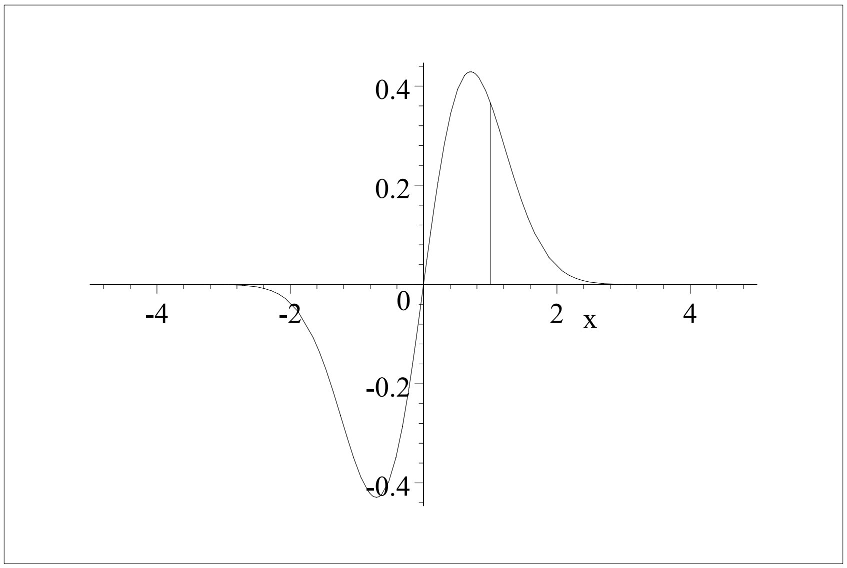
or if u **is a differentiable function of** x

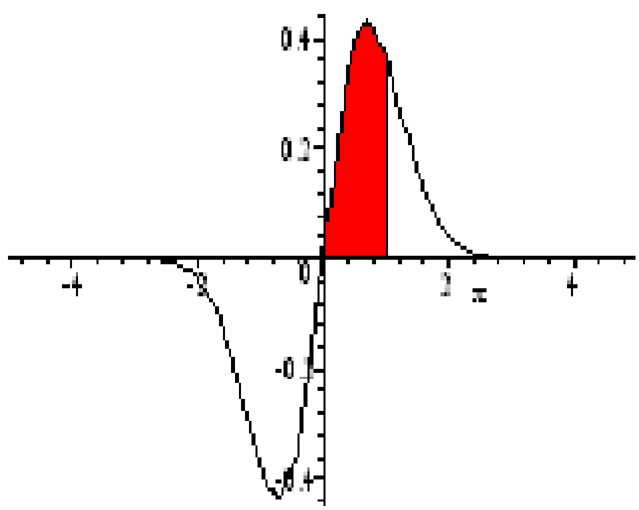
Section 5.4, please read and work on the suggested practice problems in the text

$$\frac{de^u}{dx} = \mathbf{e}^u \frac{du}{dx}$$
$$\int e^u du = e^u + C$$

Example 1:
Find the area between the graph of $y = xe^{-x^2}$ **and the x-axis from** $x = 0$ **to** $x = 1$.

$y = xe^{-x^2}$





$$\int_0^1 \mathbf{xe}^{-x^2}\mathbf{dx}$$

Note that

$\int \mathbf{xe}^{-x^2}\mathbf{dx}$

$-\mathbf{x}^2 = \mathbf{u} \rightarrow -\mathbf{2xdx} = \mathbf{du} \rightarrow \mathbf{xdx} = -\frac{1}{2}\mathbf{du}$

Section 5.4, please read and work on the suggested practice problems in the text

$\int \mathbf{xe}^{-x^2}\mathbf{dx}$

$= \int e^u\left(-\frac{1}{2}du\right)$

$= -\frac{1}{2}\int e^u du$

$= -\frac{1}{2}e^u + C$

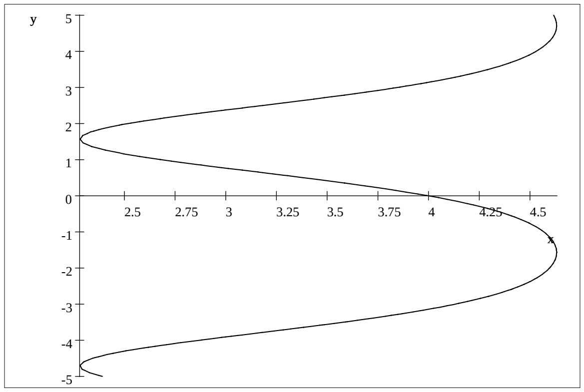
$= -\frac{1}{2}e^{-x^2} + C$

$\int_0^1 \mathbf{xe}^{-x^2}\mathbf{dx}$

$= -\frac{1}{2}e^{-x^2}\Big|_0^1$

$= -\frac{1}{2}e^{-1} + \frac{1}{2}$

Work on the section5.4 please



Find $\frac{dy}{dx} = y'$ by implicit differentiation if

$e^{\sin y} + x = 5$

wrt x

$e^{\sin y}(\cos y)\frac{dy}{dx} + 1 = 0$

$e^{\sin y}(\cos y)y' + 1 = 0$

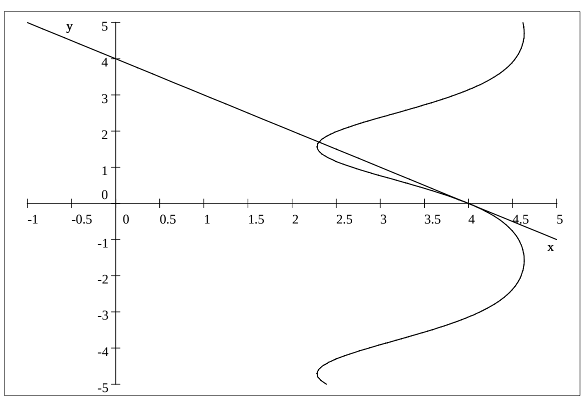
solve for y'

$e^{\sin y}(\cos y)y' = -1$

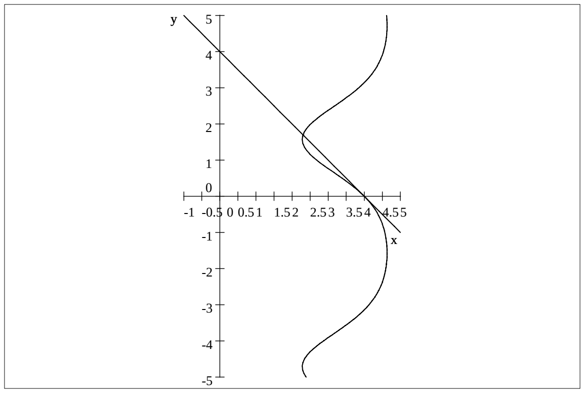
$y' = -\frac{1}{e^{\sin y}(\cos y)}$

What is the slope of the tangent line at $(4, 0)$

$y' = -\frac{1}{e^{\sin 0}(\cos 0)} = -1$



Section 5.4, please read and work on the suggested practice problems in the text



To evaluate

$$\int \frac{e^x}{e^x - 1} dx$$

set $e^x - 1 = u$

$\rightarrow e^x dx = du$

$$\int \frac{du}{u} = \ln|u| + C = \ln|e^x - 1| + C$$

....

$$\int \frac{dx}{1 + e^x}$$

Section 5.4, please read and work on the suggested practice problems in the text

Work on 5.4 and

work on

$$\int \frac{dx}{1 + e^x}$$

Another example of implicit differentiation

To use the implicit differentiation to find y’ if

$$e^x \cos y = xe^y$$

and find an equation of the tangent line to the graph at the point $(0, \frac{\pi}{2})$

$e^x \cos y = xe^y$

differentiate wrt x

$e^x \cos y + e^x(-\sin y)y' = e^y + xe^y y'$

Now solve for y'

$(-e^x \sin y - xe^y)y' = e^y - e^x \cos y$

Section 5.4, please read and work on the suggested practice problems in the text

$y' = -\frac{e^y - e^x \cos y}{e^x \sin y + xe^y}$

at $(0, \frac{\pi}{2})$

slope $-\frac{e^{\pi/2} - e^0 \cos(\pi/2)}{e^0 \sin(\pi/2) + 0e^{\pi/2}} = -e^{\frac{1}{2}\pi}$

Equation is

$y - \frac{\pi}{2} = -e^{\frac{1}{2}\pi}x$

Let us check