We are going to consider vector fields.

We may think of vector field as a function that assigns a vector to each point in a region in two or three dimensional space.

Example

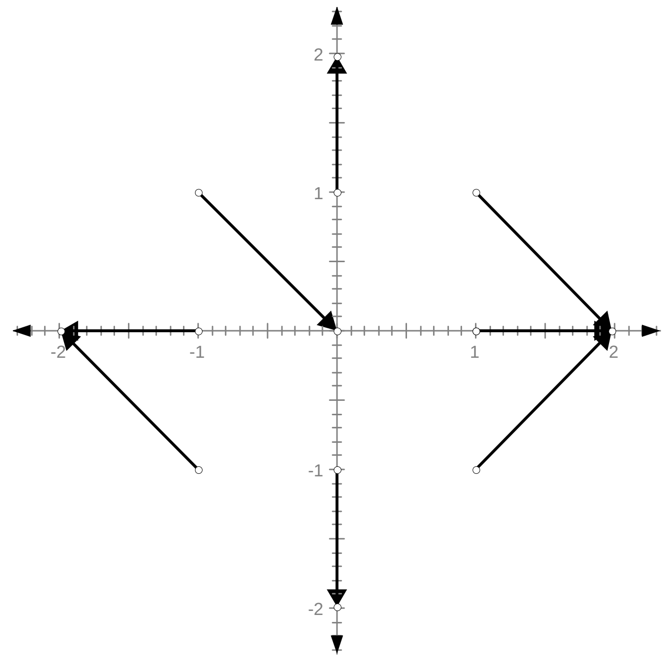
1.

$F(x,y) = xi - yj$

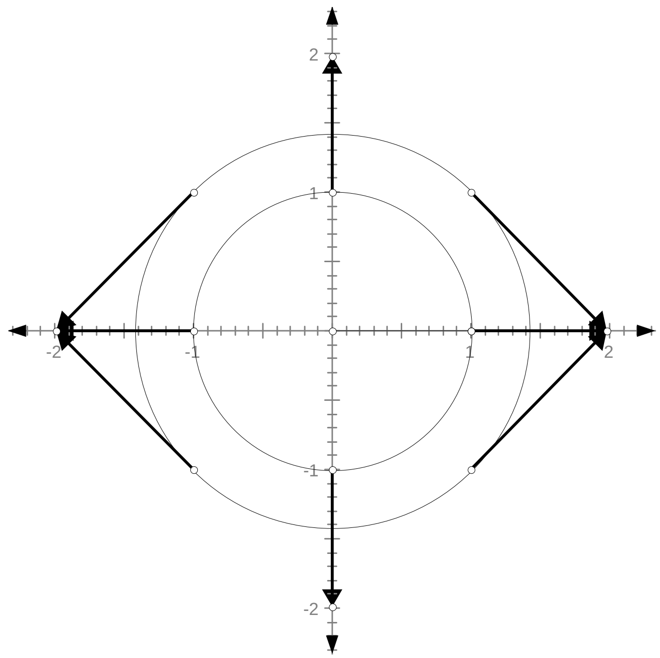
Note that $F(1,0) = i = \langle 1,0 \rangle$ $\quad F(0,1) = -j = \langle 0,-1 \rangle$ $\quad F(-1,0) = -i = \langle -1,0 \rangle$ $\quad F(0,-1) = j = \langle 0,1 \rangle$

$F(1,1) = i - j = \langle 1,-1 \rangle$ $\quad F(1,-1) = i + j = \langle 1,1 \rangle$ $\quad F(-1,1) = -i - j = \langle -1,-1 \rangle$ $\quad F(-1,-1) = -i + j = \langle -1,1 \rangle$

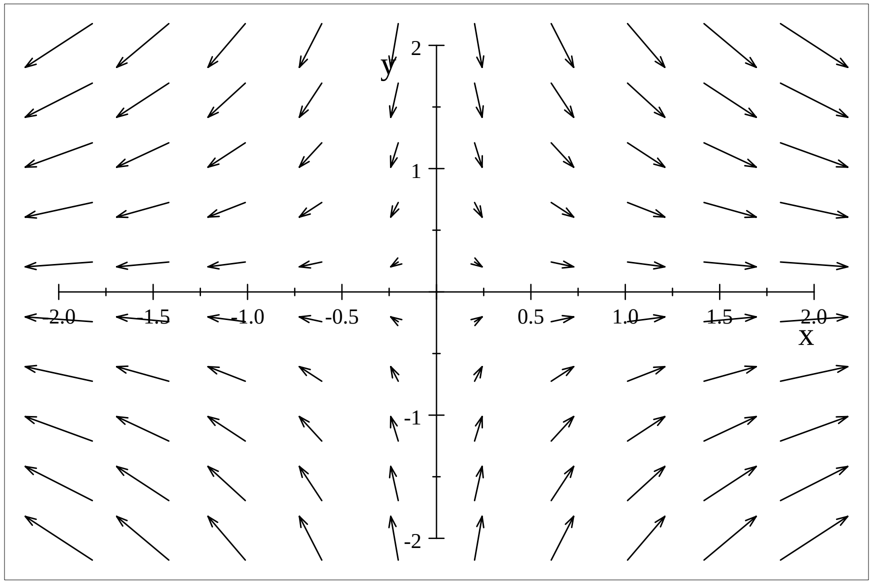
We may show the sketches of these vectors as



2
1
-1
-2
-2
-1
1
2

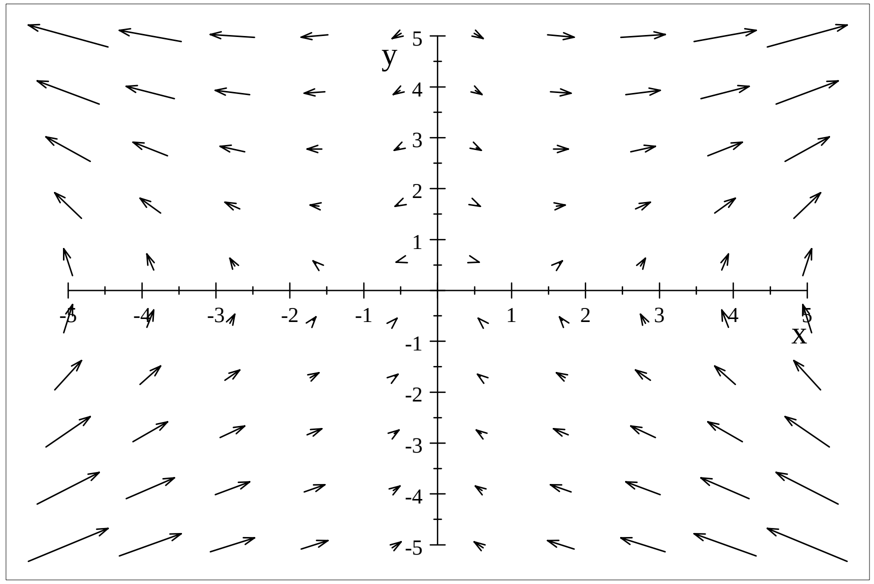


A general picture will look like (you do not have to sketch a general picture like this on the exam.)



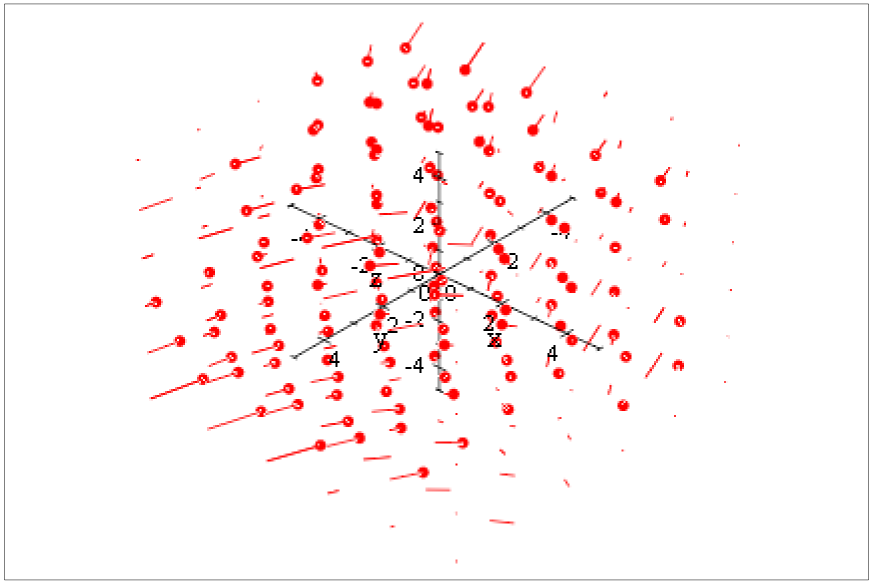
Example 2:

$$F(x,y) = 2xyi + \left(x^2 - y\right)j$$



Example 3:

$F(x,y,z) = x^2zi - 2xzj + yzk$



Note that the components of a vector field, as in this case,
$x^2z, \ -2xz, \ yzk$

are functions of x, y, z

Example 4:

∇f for a scalar function $f(x, y, z)$ is a vector field.

Definition: If $F(x,y,z) = Mi + Nj + Pk$ is a vector field such that M, N, Q have first continuous partial derivatives in an open region in the three dimensional space, then we define the Curl of the function F in the following manner

Curl$F(x,y,z) = \nabla \times F$ another notation

$$=\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)k$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Example 1

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To find $\nabla \times F$ at $(2, -1, 3)$
where $F(x,y,z) = x^2 zi - 2xzj + yzk$

$\nabla \times F$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & -2xz & yz \end{vmatrix}$$

$$=\left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(-2xz)\right)i + \left(\frac{\partial}{\partial z}\left(x^2 z\right) - \frac{\partial}{\partial x}(yz)\right)j + \left(\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}\left(x^2 z\right)\right)k$$

$$=(z + 2x)i + \left(x^2\right)j + (-2z)k$$

at $(2, -1, 3)$

Curl$F = (3 + 2(2))i + \left(2^2\right)j + (-2 \times 3)k = 7i + 4j - 6k$

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To find $\nabla \times F$ if $F(x,y,z) = e^z(yi + xj + k) = e^z yi + e^z xj + e^z k$

$\nabla \times F$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^z y & e^z x & e^z \end{vmatrix}$$

$$=\left(\frac{\partial}{\partial y}(e^{z})-\frac{\partial}{\partial z}(e^{z}x)\right)i+\left(\frac{\partial}{\partial z}(e^{z}y)-\frac{\partial}{\partial y}(e^{z})\right)j+\left(\frac{\partial}{\partial x}(e^{z}x)-\frac{\partial}{\partial y}(e^{z}y)\right)k$$
$$=-xe^{z}i+(ye^{z})j+(e^{z}-e^{z})k$$
$$=-xe^{z}i+ye^{z}j$$

Definition: A vector field F is called a conservative vector field if Curl$F=0$

Example 1:

For a differentiable scalar function $f(x,y,z)$, ∇f is a conservative vector field.

Recall that $\nabla f=\left(\frac{\partial f}{\partial x}\right)i+\left(\frac{\partial f}{\partial y}\right)j+\left(\frac{\partial f}{\partial z}\right)k$

Because

$$\nabla\times(\nabla f)$$
$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$
$$=\left(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\right)i+\left(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z}\right)j+\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)k$$
$$=0$$

Example2:

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Given that $F = y^2z^3 i + 2xyz^3 j + 3xy^2z^2 k$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$
$$= \left(\frac{\partial}{\partial y}\left(3xy^2z^2\right) - \frac{\partial}{\partial z}\left(2xyz^3\right) \right) i + \left(\frac{\partial}{\partial z}\left(y^2z^3\right) - \frac{\partial}{\partial x}\left(3xy^2z^2\right) \right) j + \left(\frac{\partial}{\partial x}\left(2xyz^3\right) - \frac{\partial}{\partial y}\left(y^2z^3\right) \right) k$$
$$= \left(6xyz^2 - 6xyz^2\right) i + \left(3y^2z^2 - 3y^2z^2\right) j + \left(2yz^3 - 2yz^3\right) k$$
$$= 0$$

In fact $\nabla \times F = 0$ if and only if we can find a scalar function f such that $F = \nabla f$

In such a case, f is called the potential function of F

In the above example, we noted that for $F = y^2z^3 i + 2xyz^3 j + 3xy^2z^2 k$

$\nabla \times F = 0$

therefore we should be able to find a potential function f for F

so that $F = \nabla f$

Or

$$y^2z^3 i + 2xyz^3 j + 3xy^2z^2 k = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

That is

$$\frac{\partial f}{\partial x} = y^2z^3 \qquad \frac{\partial f}{\partial y} = 2xyz^3 \qquad \frac{\partial f}{\partial z} = 3xy^2z^2$$

Please pay attention to the following methodology to obtain the function f from the above information
Consider

$$\frac{\partial f}{\partial x} = y^2z^3$$

Integrate with respect to x , treating y and z as constants

$f = xy^2z^3 + g(y,z) + K_1$, where $g(y,z)$ is a function of y and z only and K_1 is an absolute constant

Similarly

$$\frac{\partial f}{\partial y} = 2xyz^3 \rightarrow f = xy^2z^3 + h(z,x) + K_2$$
$$\frac{\partial f}{\partial z} = 3xy^2z^2 \rightarrow f = xy^2z^3 + k(x,y) + K_3$$

compare

$$f = xy^2z^3 + g(y,z) + K_1$$
$$f = xy^2z^3 + h(z,x) + K_2$$
$$f = xy^2z^3 + k(x,y) + K_3$$

$f(x,y,z) = xy^2z^3 + K$ where K is an absolute constant.

We may check our answer by verifying that

$$\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = F$$

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To check if $F = \frac{x}{x^2+y^2} i + \frac{y}{x^2+y^2} j + k$ is conservative.

In case it is conservative, to find f such that $F = \nabla f$

$$\nabla \times F$$
$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 1 \end{vmatrix}$$
$$= \left(\frac{\partial(1)}{\partial y} - \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2}\right)\right) i + \left(\frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial(1)}{\partial x}\right) j + \left(\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right) k$$
$$=(0)i + (0)j + \left(-\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right) k$$
$$=0$$

Therefore F is conservative

To find f such that $F = \nabla f$

$$\frac{x}{x^2+y^2} i + \frac{y}{x^2+y^2} j + k = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Now

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2} \rightarrow f = \frac{1}{2}\ln\left(x^2+y^2\right) + g(y,z) + K_1$$ USED $$\boxed{\int \frac{u}{u^2+a^2}du = \frac{1}{2}\ln\left(u^2+a^2\right)}$$

$$\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2} \rightarrow f = \frac{1}{2}\ln\left(x^2+y^2\right) + h(z,x) + K_2$$

$$\frac{\partial f}{\partial z} = 1 \rightarrow f = z + k(x,y) + K_3$$

Comparing

$f = \frac{1}{2}\ln\left(x^2+y^2\right) + g(y,z) + K_1 \qquad f = \frac{1}{2}\ln\left(x^2+y^2\right) + h(z,x) + K_2 \qquad f = z + k(x,y) + K_3$

We have

$$f(x,y,z) = \frac{1}{2}\ln\left(x^2+y^2\right) + z + K$$

To check the answer, compute ∇f and check if it is actually F

In case, we have a vector field in two dimensions

$$F(x,y) = Mi + Nj$$

We shall call F conservative if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

In case $F(x,y) = Mi + Nj$ is conservative, we should be able to find f such that
$F = \nabla f$

#38 on the page 1063

$F(x,y) = 3x^2y^2i + 2x^3yj$

$M = 3x^2y^2$
$N = 2x^3y$

$\frac{\partial M}{\partial y} = 6x^2y$
$\frac{\partial N}{\partial x} = 6x^2y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

therefore the field F is conservative

To find f such that $F = \nabla f$

That is

$$\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j = 3x^2y^2i + 2x^3yj$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 \rightarrow f = x^3y^2 + g(y) + K_1$$
$$\frac{\partial f}{\partial y} = 2x^3y \rightarrow f = x^3y^2 + h(x) + K_2$$

The compariosn gives that

$$f(x,y) = x^3y^2 + K$$

....

Line Integral

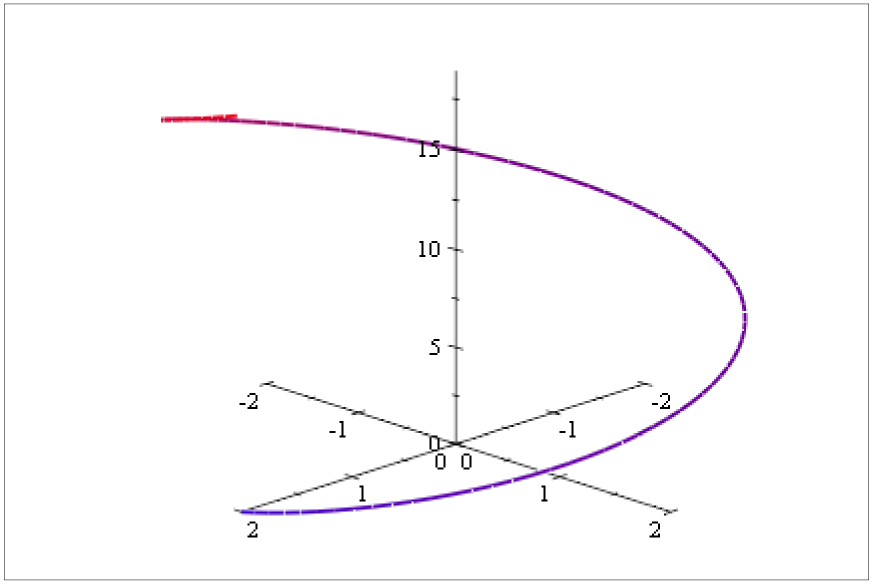
Let a smooth curve C be defined by the parametrization $x = x(t), y = y(t), z = z(t)$, $a \leq t \leq b$

then the line integral $\int_C f ds = \int_a^b f(x(t),y(t),z(t)) \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\, dt$

A good application of such a line integral is to obtain the mass of a wire with given density

#26 on the page 1075

To find the mass of the wire that is given by $\vec{r}(t) = 2(\cos t)i + 2(\sin t)j + 3tk$, $0 \le t \le 2\pi$
where the density of the wrire is $\rho(x,y,z) = k + z$



The mass

$$\int_C \rho ds$$

$$= \int_0^{2\pi} (k+z) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} (k+3t) \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (3)^2}\, dt$$

$$= \int_0^{2\pi} (k+3t) \sqrt{4\sin^2 t + 4\cos^2 t + 9}\, dt$$

$$= \int_0^{2\pi} (k+3t) \sqrt{4\left(\sin^2 t + \cos^2 t\right) + 9}\, dt$$

$$= \int_0^{2\pi} (k+3t) \sqrt{4+9}\, dt$$

$$= \int_0^{2\pi} (k+3t) \sqrt{13}\, dt$$

$$= \sqrt{13} \int_0^{2\pi} (k+3t) dt$$

$$=\sqrt{13}\left(\left(kt+\frac{3t^2}{2}\right)\Big|_0^{2\pi}\right)$$
$$=\sqrt{13}\left[\left(2\pi k+\frac{3(2\pi)^2}{2}\right)-\left(k(0)+\frac{3(0)^2}{2}\right)\right]$$
$$=\sqrt{13}\left(2\pi k+6\pi^2\right)$$

The line integral of a vector field

Let F be a continuous vector field defined on a smooth curve C that is given by $\vec{r}(t)$, $a \le t \le b$
The line integral of F on C is given by

$$\int_C F \cdot dr = \int_C F \cdot T ds = \int_a^b F(x(t),y(t),z(t)) \cdot \vec{r}\,'(t)dt$$

An application of such a line integral may be seen in the computation of work done by a force.

Example:

To compute the work done by the force field $F = xyi + yzj + zxk$
in moving a particle along $x = t \quad y = t^2 \quad z = t^3 \qquad 0 \le t \le 1$

here $\vec{r}(t) = ti + t^2 j + t^3 k$

$F(x(t), y(t), z(t)) = (t)\left(t^2\right)i + \left(t^2\right)\left(t^3\right)j + (t)\left(t^3\right)k = t^3 i + t^5 j + t^4 k$

$\vec{r}\,'(t) = i + 2tj + 3t^2 k$

$F \cdot \vec{r}\,' = t^3 + t^5(2t) + t^4\left(3t^2\right) = t^3 + 2t^6 + 3t^6 = t^3 + 5t^6$

$$\int_0^1 F \cdot \vec{r}\,' dt = \int_0^1 \left(t^3 + 5t^6\right)dt = \left(\frac{t^4}{4} + \frac{5t^7}{7}\Big|_0^1\right) = \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$$

#40 on the page 1076

To find the work done by the force $F(x, y, z) = yzi + xzj + xyk$

in moving a particle along the straight line from $(0, 0, 0)$ to $(5, 3, 2)$

Note that an equation of the line connecting $(0, 0, 0)$ to $(5, 3, 2)$

is $x = 5t \qquad y = 3t \qquad z = 2t$

with $t = 0$ to $t = 1$

$\vec{r}(t) = 5ti + 3tj + 2tk$
$\vec{r}\,'(t) = 5i + 3j + 2k$

$$F(x(t), y(t), z(t)) = (3t)(2t)i + (5t)(2t)j + (5t)(3t)k = 6t^2 i + 10t^2 j + 15t^2 k$$

$$F \cdot \vec{r}\,' = 30t^2 + 30t^2 + 30t^2 = 90t^2$$

$$\int_0^1 F \cdot \vec{r}\,' dt = \int_0^1 90t^2 dt = \left(30t^3 \Big|_0^1 \right) = 30$$

....

Note that if F is conservative, that is $F = \nabla f$ for some scalar function f

then

For any path C from $t = a$ to $t = b$ given by $\vec{r}(t)$

$$F \cdot \vec{r}' = \nabla f \cdot \vec{r}' = \left(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right) \cdot \left(\frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k\right) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \frac{df}{dt}$$

and

$$\int_C F \cdot T ds = \int_P^Q \frac{df}{dt} dt = f(Q) - f(P)$$

That is to say, the line intergral just depends on the values of the potential function at the end points and is consequently independent of path for a conservative vector field F

and also For any closed curve C, $\int_C F \cdot T ds = 0$ IF F is conservative

#20 on the page 1086

To find $\int_C F \cdot dr$

where $F = i + zj + yk$

and C is

a) $\vec{r}(t) = (\cos t)i + (\sin t)j + t^2 k \qquad 0 \le t \le \pi$
b) $\vec{r}(t) = (1-2t)i + \pi^2 tk \qquad 0 \le t \le 1$

Note that

$$\text{Curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & z & y \end{vmatrix} = \left(\frac{\partial y}{\partial y} - \frac{\partial z}{\partial z}\right)i + \left(\frac{\partial 1}{\partial z} - \frac{\partial y}{\partial x}\right)j + \left(\frac{\partial z}{\partial x} - \frac{\partial 1}{\partial y}\right)k = 0$$

therefore F is conservative and we can find the value of the line integral just by evaluating the potential function at the end points

Let us find a function f such that $\nabla f = F$

that is

$$\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = i + zj + yk$$

$$\frac{\partial f}{\partial x} = 1 \rightarrow f = x + g(y,z) + K_1$$

$$\frac{\partial f}{\partial y} = z \rightarrow f = yz + h(z,x) + K_2$$

$$\frac{\partial f}{\partial z} = y \rightarrow f = yz + k(x,y) + K_3$$

Compare the three values to see that

$$f(x,y,z) = x + yz + K$$

Therefore

for

a) $\vec{r}(t) = (\cos t)i + (\sin t)j + t^2 k \qquad 0 \leq t \leq \pi$

at $t = 0,\ (x,y,z) = (1,0,1)$
at $t = \pi,\ (x,y,z) = \left(-1,0,\pi^2\right)$

$$\int_0^{\pi} F \cdot dr = f\left(-1, 0, \pi^2\right) - f(1, 0, 1) = (-1 + 0) - (1 + 0) = -2$$

b)

$\vec{r}(t) = (1 - 2t)i + \pi^2 tk \qquad\qquad 0 \le t \le 1$

$x = 1 - 2t \qquad y = 0 \qquad z = \pi^2 t$

when $t = 0 \qquad (x, y, z) = (1, 0, 0)$
when $t = 1 \qquad (x, y, z) = \left(-1, 0, \pi^2\right)$

$$f\left(-1, 0, \pi^2\right) - f(1, 0, 0) = (-1) - (1) = -2$$

#22 on the page 1087

To find $\int_C F \cdot dr$

where $F = -yi + xj + 3xz^2k$

and C is given by

a) $\vec{r}(t) = (\cos t)i + (\sin t)j + tk \qquad 0 \le t \le \pi$

b) $\vec{r}(t) = (1-2t)i + \pi tk \qquad 0 \le t \le 1$

In this Curl$F \ne 0$

therefore we are going to work it out the routine way

a)

$$F(x(t), y(t), z(t)) = -(\sin t)i + (\cos t)j + 3(\cos t)\left(t^2\right)k = -(\sin t)i + (\cos t)j + 3t^2(\cos t)k$$

$$\frac{d\vec{r}}{dt} = -(\sin t)i + (\cos t)j + k$$

$$\int_0^{\pi} F \cdot dr$$

$$= \int_0^{\pi} F \cdot \frac{dr}{dt} dt$$

$$= \int_0^{\pi} \left(-(\sin t)i + (\cos t)j + 3t^2(\cos t)k\right) \cdot (-(\sin t)i + (\cos t)j + k) dt$$

$$=\int_0^{\pi}\left(\sin^2 t+\cos^2 t+3t^2\cos t\right)dt$$

$$=\int_0^{\pi}\left(1+3t^2\cos t\right)dt$$

$$\int_0^{\pi} 1dt = \pi$$

Use the integration by parts (may use the short cut of the tabular integration)

to find

$$\int 3t^2\cos t dt = 3t^2\sin t+6t\cos t-6\sin t+C$$

$$\int_0^{\pi} 3t^2\cos t dt = \left(3t^2\sin t+6t\cos t-6\sin t\Big|_0^{\pi}\right) = \left(3\pi^2\sin\pi+6\pi\cos\pi-6\sin\pi\right)-\left(3(0)^2\sin 0+6(0)\cos 0-6\sin 0\right) = -6\pi$$

Therefore

$$\int_0^{\pi} \left(1 + 3t^2 \cos t\right) dt = \pi - 6\pi = -5\pi$$

b)

$\vec{r}(t) = (1-2t)i + \pi t k \qquad\qquad 0 \le t \le 1$

$x = (1-2t) \qquad y = 0 \qquad z = \pi t$

$\vec{r}'(t) = -2i + \pi k$

$F(x(t), y(t), z(t)) = 0i + (1-2t)j + 3(1-2t)(\pi t)^2 k$

$F \cdot \vec{r}' = 3\pi(1-2t)(\pi t)^2 = 3\pi^3 t^2 (1-2t) = 3\pi^3 \left(t^2 - 2t^3\right)$

$$\int_C F \cdot dr = \int_0^1 3\pi^3 \left(t^2 - 2t^3\right) dt = 3\pi^3 \left(\frac{t^3}{3} - \frac{t^4}{2} \Big|_0^1\right) = 3\pi^3 \left(\frac{1}{3} - \frac{1}{2}\right) = -\frac{\pi^3}{2}$$

Green's Theorem relates the line integral to a double integral in the following circumstance
(Read the page 1089 for more details)

If we have a region S that is bounded by ONE simple closed curve C (a curve that does not cross itself) and the boundary curve is piecewize smooth and M and N have continuous partial derivatives on S and C then

$$\int_C (Mdx + Ndy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

#16 on the page 1096

To evaluate $\int_C (e^x \cos 2y dx - 2e^x \sin 2y dy)$ along the boundary of the circle $x^2 + y^2 = a^2$

In this case $M = e^x \cos 2y$ and $N = -2e^x \sin 2y$

Note that

$$\frac{\partial N}{\partial x} = -2e^x \sin 2y$$

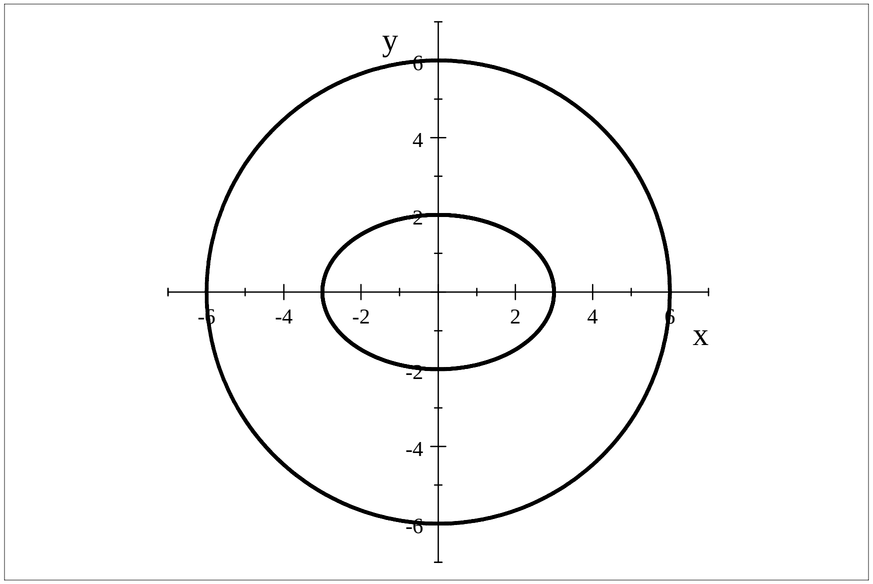
$$\frac{\partial M}{\partial y} = -2e^x \sin 2y$$

therefore $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

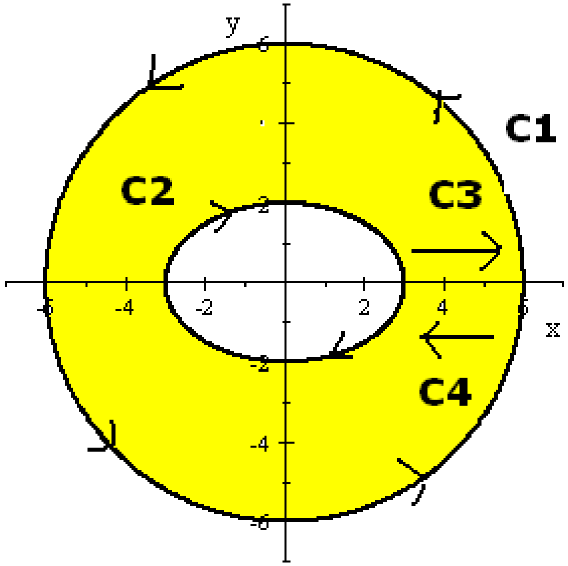
$$\int_C (Mdx + Ndy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_S 0 dA = 0$$

#18 To evaluate $\int_C \left(e^{-x^2/2} - y\right)dx + \left(e^{-y^2/2} + x\right)dy$

where C is the boundary of the region between the graphs of the circle $x = 6\cos\theta$, $y = 6\sin\theta$ and the ellipse $x = 3\cos\theta$, $y = 2\sin\theta$



If treat C as C1+C4+C2+C3 as shown below



then we can apply the Green's Theorem to evaluate

$$\int_C \left(e^{-x^2/2} - y\right)dx + \left(e^{-y^2/2} + x\right)dy$$

Here $M = e^{-x^2/2} - y$ $\qquad N = e^{-y^2/2} + x$

$\dfrac{\partial M}{\partial y} = -1 \qquad \dfrac{\partial N}{\partial x} = 1$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 2$$

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_S (2) dA = 2 \iint_S dA$$

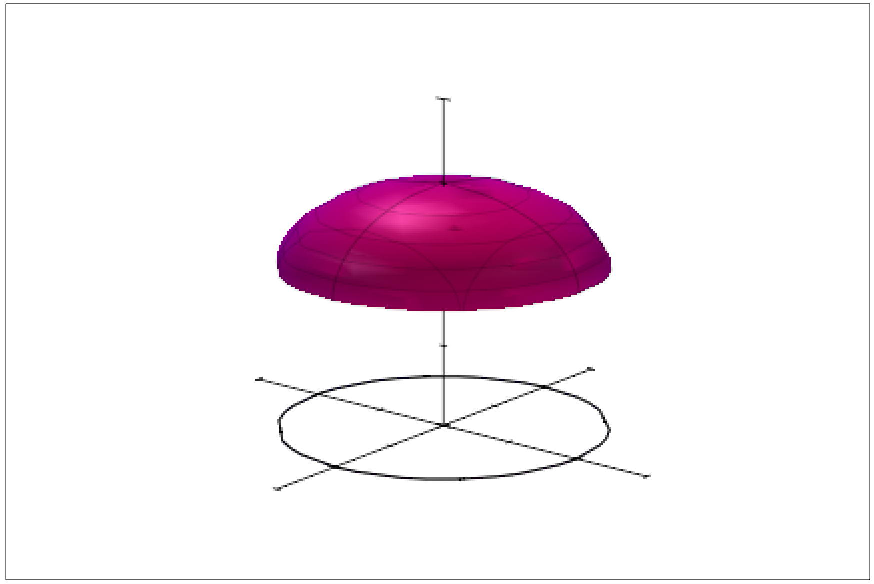
$\iint_S dA$ is the area of the region between the circle and the ellipse as shown above

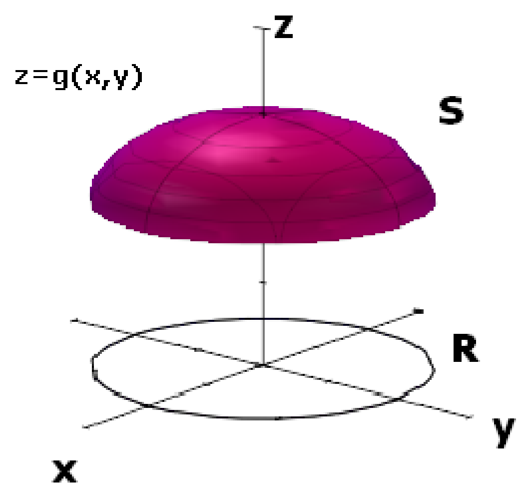
Circle of radius 6 has area $\pi(6)^2 = 36\pi$

Ellipse with semimajor axis 3 and semininor axis 2 has area $\pi(3)(2) = 6\pi$

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = 2(36\pi - 6\pi) = 60\pi$$

Evaluation of surface Integrals




z
z=g(x,y)
S
R
y
x

In the above picture the surface S has equation $z = g(x,y)$
the projection of the surface S in the xy-plane is R
$g, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$ are continuous on R

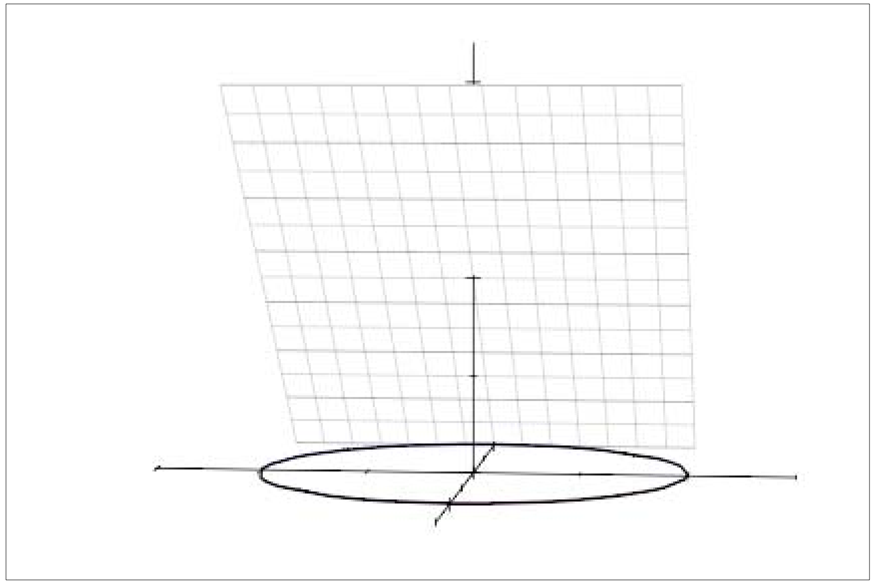
and f is continuous on S

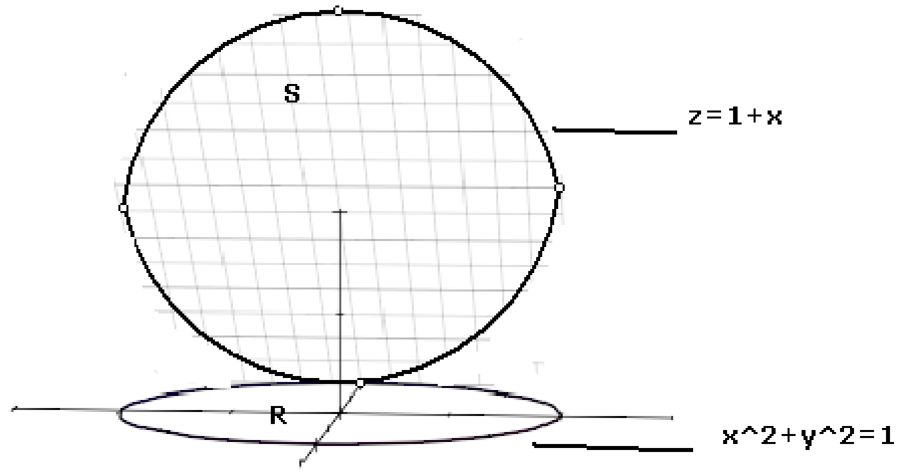
then

the surface integral

$$\iint_S f dS = \iint_R f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

Example:
To evaluate $\iint_S z ds$ where S is the part of the plane $z = 1 + x$ that lies directly above the unit disk $x^2 + y^2 = 1$





In this case

$$\iint_S z\,dS$$

$$= \iint_R (1+x)\sqrt{1+\left(\frac{\partial(1+x)}{\partial x}\right)^2+\left(\frac{\partial(1+x)}{\partial y}\right)^2}\,dA$$

$$= \iint_R (1+x)\sqrt{1+(1)^2+(0)^2}\,dA$$

$$= \iint_R (1+x)\sqrt{2}\,dA$$

$$=\sqrt{2}\iint_R (1+x)\,dA$$

Since R is the region bounded by the unit circle $x^2+y^2=1$

transforming to polar coordinates should help

$$\iint_S z\,dS$$

$$=\sqrt{2}\iint_R (1+x)\,dA$$

$$=\sqrt{2}\int_0^{2\pi}\int_0^1 (1+r\cos\theta)\,r\,dr\,d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\int_0^1 \left(r+r^2\cos\theta\right)dr\,d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\left(\frac{r^2}{2}+\frac{r^3}{3}\cos\theta\Big|_0^1\right)d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{3}\cos\theta\right)d\theta$$

$$=\sqrt{2}\left(\frac{\theta}{2}+\frac{1}{3}\sin\theta\Big|_0^{2\pi}\right)$$

$$=\sqrt{2}\,\pi$$

An important application of the surface integral is mentioned on the page 1114 of the text that if a fluid is moving through a surface S with continuous velocity given by F then the volume of the fluid crossing the surface S per unit time is called the flux of F across S and is evaluated by $\iint\limits_S F \cdot N dS$

where S is oriented by the unit normal vector N (check page 1113)

We call a surface S orientable if it is possible to choose a unit normal vector N at each point of the surface that varies continuously over S

If an orientable surface is given by $z = g(x,y)$ which is a level surface of $G(x,y,z) = z - g(x,y)$
then recall that ∇G is a vector normal to $G(x,y,z) = 0$ or $z = g(x,y)$

therefore we can use $\frac{\nabla G}{||\nabla G||}$ OR $-\frac{\nabla G}{||\nabla G||}$ to orient the surface S

$$\nabla G = -\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k$$

$$||\nabla G|| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

$$N = \frac{\nabla G}{||\nabla G||} = \frac{\left(-\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k\right)}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

$$\iint_S F \cdot N dS = \iint_R F \cdot \frac{\left(-\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k\right)}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}} \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA = \iint_R F \cdot \left(-\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k\right) dA = \iint_R F \cdot \nabla G dA$$

Example:

To evaluate the flux of $F = xi + yj + 2zk$ through S where S is given by $z = \sqrt{9 - x^2 - y^2}$

First, we should find the unit normal vector N

Take $G(x, y, z) = z - \sqrt{9 - x^2 - y^2}$

$$\nabla G = \frac{\partial G}{\partial x} i + \frac{\partial G}{\partial y} j + \frac{\partial G}{\partial z} k$$

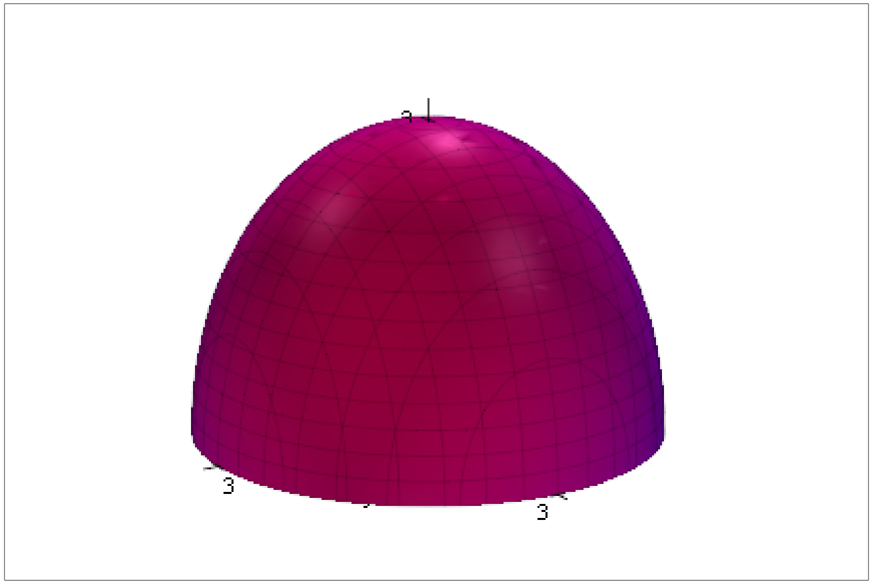
that is

$$\nabla G = \frac{x}{\sqrt{9 - x^2 - y^2}} i + \frac{y}{\sqrt{9 - x^2 - y^2}} j + k$$

on the surface S

$$F = xi + yj + 2zk = xi + yj + 2\sqrt{9 - x^2 - y^2}\, k$$

$$F \cdot G = \frac{x^2}{\sqrt{9 - x^2 - y^2}} + \frac{y^2}{\sqrt{9 - x^2 - y^2}} + 2\sqrt{9 - x^2 - y^2} = \frac{x^2 + y^2 + 2(9 - x^2 - y^2)}{\sqrt{9 - x^2 - y^2}} = \frac{18 - x^2 - y^2}{\sqrt{9 - x^2 - y^2}}$$



The projection R of S in the xy-plane is the circular disk bounded by $x^2 + y^2 = 9$

$$\iint_S F \cdot N dS$$

$$= \iint_R F \cdot \nabla G dA$$

$$= \iint_R \frac{18 - x^2 - y^2}{\sqrt{9 - x^2 - y^2}} dA$$

$$= \int_0^{2\pi} \int_0^3 \frac{18 - r^2}{\sqrt{9 - r^2}} r dr d\theta$$

For the evaluation of

$$\int_0^3 \frac{18-r^2}{\sqrt{9-r^2}} r\,dr$$

let $\sqrt{9-r^2} = u \to 9 - r^2 = u^2 \to -rdr = udu \to rdr = -udu$

$$\int_0^3 \frac{18-r^2}{\sqrt{9-r^2}} r\,dr$$

$$= -\int_3^0 \frac{18-(9-u^2)}{u} u\,du$$

$$= \int_0^3 \left(9 + u^2\right) du$$

$$= 9u + \frac{u^3}{3} \Big|_0^3$$

$$= 36$$

$$\int_0^{2\pi} \int_0^3 \frac{18-r^2}{\sqrt{9-r^2}} r\,dr\,d\theta = \int_0^{2\pi} (36)\,d\theta = 72\pi$$

Divergence of a vector field

For a vector field $F(x,y,z) = F_1 i + F_2 j + F_3 k$

we define the Divergence of F

denoted by div$F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Example

For $F(x,y,z) = x^2 yi + yzj + e^{-x} z^2 k$

div$F = 2xy + z + 2ze^{-x^2}$

If we consider a fluid motion with the velocity F
then div$F$ measures the tendency of the fluid to diverge from the point (x,y,z)

The Divergence Theorem is a good tool to evaluate the surface integral by transforning it to a volume integral

For a region Q bounded by a closed surface S

we have $\boxed{\iint_S F \cdot N dS = \iiint_Q (\operatorname{div} F) dV}$

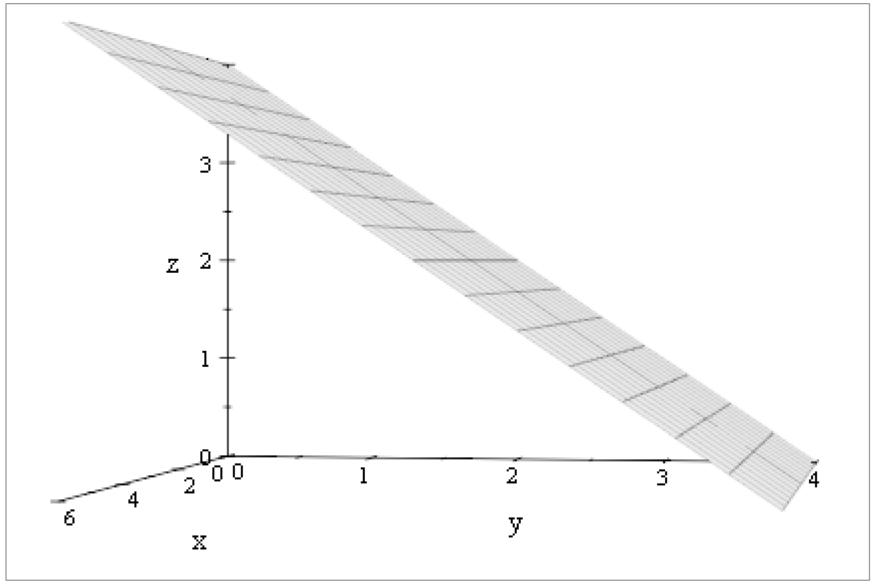
under the conditions stated in the Theorem 15.12 on the page 1120 of the text

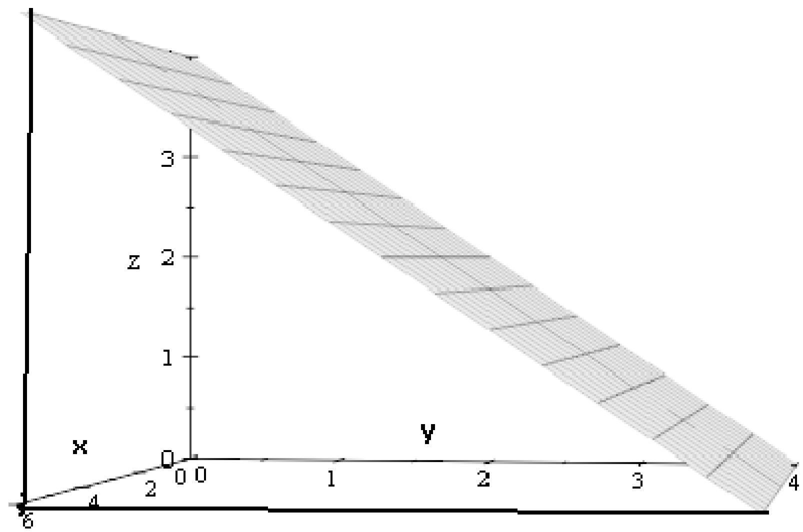
#14 on the page 1127

To use the divergence theorem to evaluate $\iint_S F \cdot N dS$

where $F(x, y, z) = xe^z i + ye^z j + e^z k$

and S is $z = 4 - y, z = 0, x = 0, x = 6, y = 0$


z
3
2
1
0
x
6
4
2
0
y
0
1
2
3
4


z
3
2
1
0
x
6
4
2
0
y
0
1
2
3
4

$$F(x,y,z) = xe^z i + ye^z j + e^z k$$
$$\text{div}F = e^z + e^z + e^z = 3e^z$$

$$\iint_S F \cdot N dS = \iiint_Q \text{div}F dV = \iiint_Q 3e^z dV$$

$$\iiint_Q 3e^z dV$$
$$= \int_0^6 \int_0^4 \int_0^{4-y} 3e^z dz dy dx$$
$$= 3\int_0^6 \int_0^4 \left(e^z \Big|_0^{4-y} \right) dy dx$$
$$= 3\int_0^6 \int_0^4 \left(e^{4-y} - 1 \right) dy dx$$
$$=3\int_0^6 \left(-e^{4-y} - y \Big|_0^4 \right) dx$$
$$=3\int_0^6 \left(-1 - 4 + e^4 \right) dx$$

$$=3\int_0^6 \left(e^4 - 5\right)dx$$

$$=3\left(e^4 - 5\right)6$$

$$= 18\left(e^4 - 5\right)$$

#16 on the page 1127

To use the divergence theorem to evaluate $\iint_S F \cdot NdS$

where $F(x,y,z) = 2(xi + yj + zk)$

and S is $z = \sqrt{4 - x^2 - y^2}$, $z = 0$

In this case Q is the upper hemispherical region

div$F = 2 + 2 + 2 = 6$

$$\iint_S F \cdot NdS = \iiint_Q \text{div}FdV = \iiint_Q 6dV = 6\iiint_Q dV$$

$\iiint\limits_Q dV$ is the volume enclosed in a hemisphere of rdaius 2

which is $\frac{1}{2}\left(\frac{4}{3}\pi(2)^3\right) = \frac{16\pi}{3}$

therefore $6\iiint\limits_Q dV = 16\left(\frac{16\pi}{3}\right) = 32\pi$

3-D version of Green's Theorem

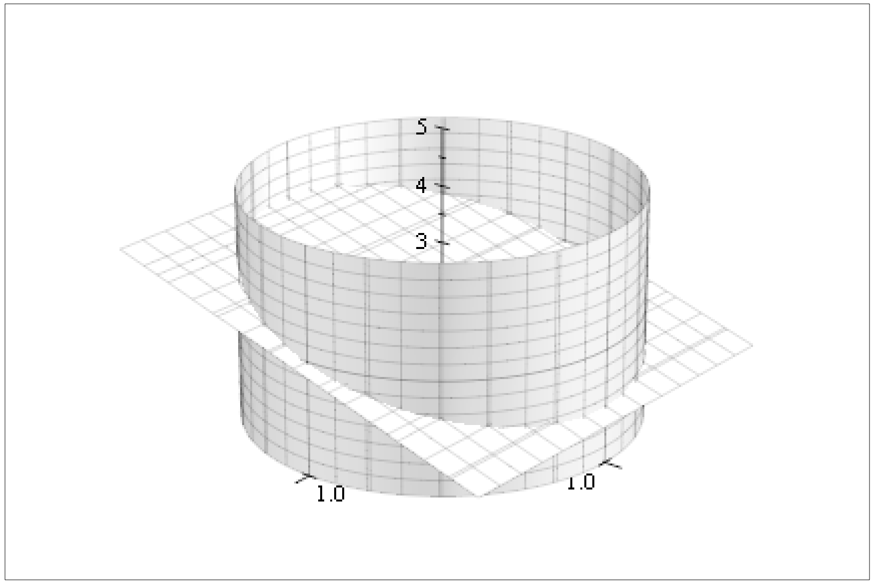
Stoke's Theorem

$$\int\limits_C F \cdot dr = \iint\limits_S (\text{Curl}F) \cdot N dS$$

under the conditions stated in the Thorem 15.13 on the page 1128

To evaluate $\int\limits_C \left(-y^2 i + xj + z^2 k\right) \cdot dr$

where C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$
(assume a counter clockwise orientation for the curve C)



Stoke's THeorem gives

$$\int_C \left(-y^2 i + xj + z^2 k\right) \cdot dr = \iint_S \left(\text{Curl}\left(-y^2 i + xj + z^2 k\right)\right) \cdot N dS$$

$$\text{Curl}\left(-y^2 i + xj + z^2 k\right) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \left(\frac{\partial z^2}{\partial y} - \frac{\partial x}{\partial z}\right)i + \left(\frac{\partial(-y^2)}{\partial z} - \frac{\partial z^2}{\partial x}\right)j + \left(\frac{\partial x}{\partial x} + \frac{\partial y^2}{\partial y}\right)k = (1+2y)k$$

$$\iint_S ((1+2y)k) \cdot N dS$$

The surface is given by $y + z = 2$, take $G = y + z - 2$

$$\iint_R ((1+2y)k) \cdot \nabla G dA$$ (R is the projection of the surface in the xy-plane which is the disk $x^2 + y^2 \leq 1$)

$$= \iint_R ((1+2y)k) \cdot (j+k) dA$$

$$= \iint_R (1+2y) dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(r + 2r^2 \sin\theta\right) dr d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} + \frac{2r^3}{3}\sin\theta \Big|_0^1\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3} \sin\theta \right) d\theta$$

$$= \frac{\theta}{2} - \frac{2}{3} \cos\theta \Big|_0^{2\pi}$$

$$= \pi - \frac{2}{3} \cos 2\pi - 0 + \frac{2}{3} \cos 0$$

$$= \pi$$

(could note that $\int_0^{2\pi} \sin\theta d\theta = 0$)