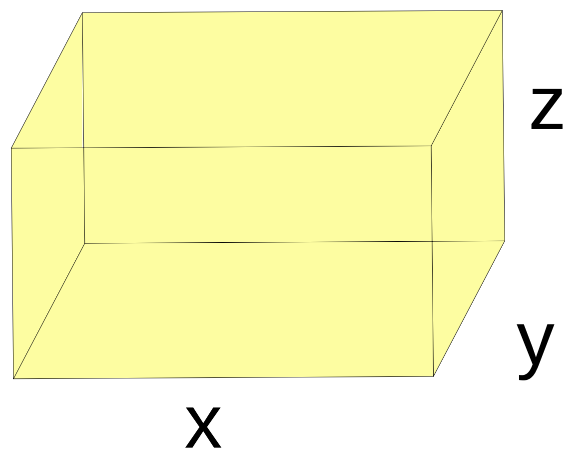
Using double integrals to compute the volume of a variety of regions

We know that the volume of a rectangular box



is xyz

Please read the section 14.2 and understand the description of the division of the region under the graph of $f(x,y)$ into prizms and then taking the limit of the corresponding Riemann Sum to define $\iint\limits_R f(x,y)dA$ and the use of the value of $\iint\limits_R f(x,y)dA$ (if it exists) to the volume of the region that lies above R and below the graph of $f(x,y)$, where $f(x,y) \geq 0$

It will be very helpful if you read and understand the pags 990 thru 997, in case any step is not clear, kindly ask me, I shall try to calrify ASAP.

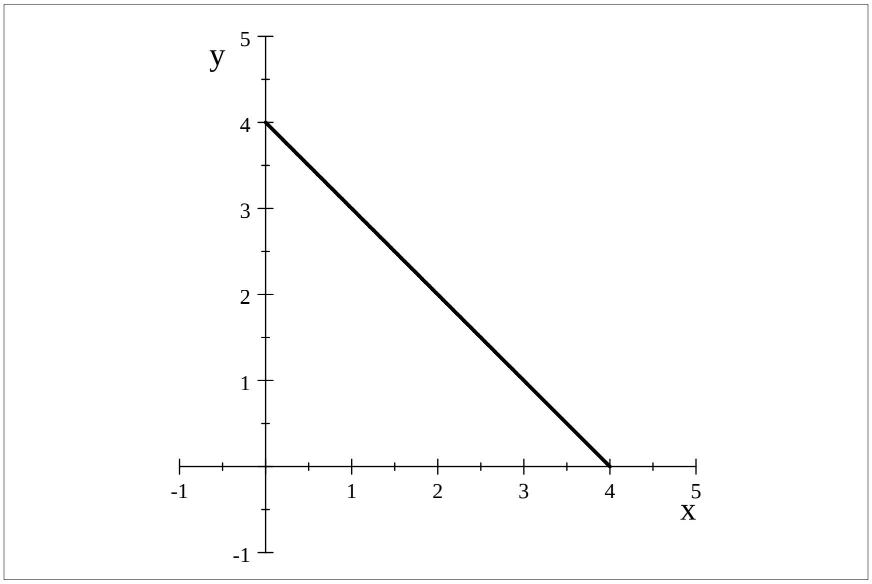
Let us continue with the process through examples:

We shall use exercises in the section 14.2

#16

To set up and evaluate $\iint_R xe^y dA$, where R is the triangle bounded by $y = 4 - x, y = 0, x = 0$

$4 - x$

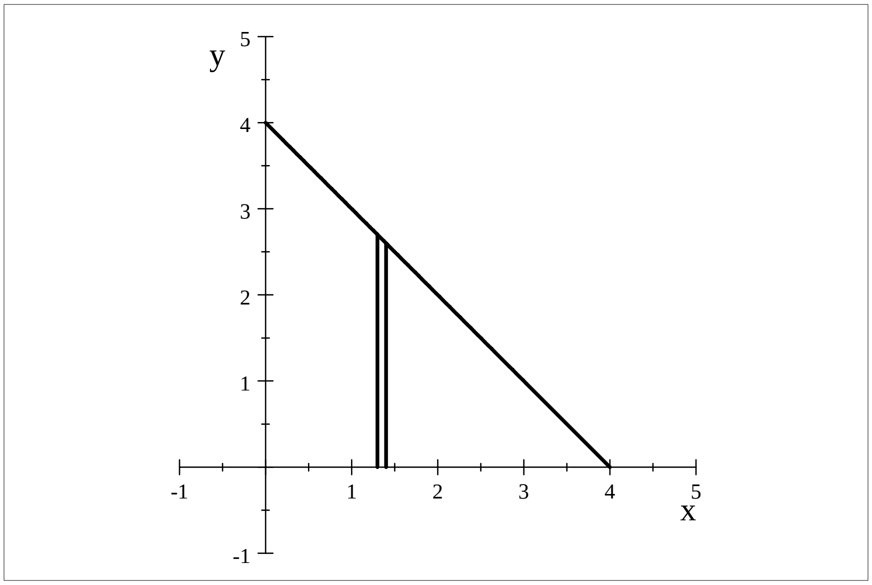


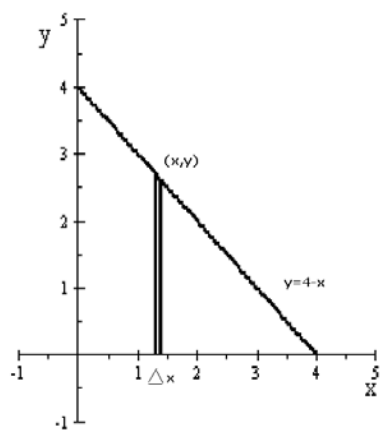
For the integration of xe^y , note that if we integrate wrt x first,

we shall have $\frac{x^2}{2}e^y$

on the other hand if we integrate wrt y first,

we shall have xe^y which should be easier to deal with





$$\iint_R xe^y dA = \int_0^4 \int_0^{4-x} xe^y dydx$$

$$\int_0^4 \int_0^{4-x} xe^y dydx$$

$$= \int_0^4 \left(\int_0^{4-x} xe^y dy \right) dx$$

$$= \int_0^4 (xe^y|_0^{4-x}) dx$$

$$=\int_0^4 (xe^{4-x} - xe^0)dx$$
$$=\int_0^4 (xe^{4-x} - x)dx$$
$$=\int_0^4 xe^{4-x}dx - \int_0^4 xdx$$

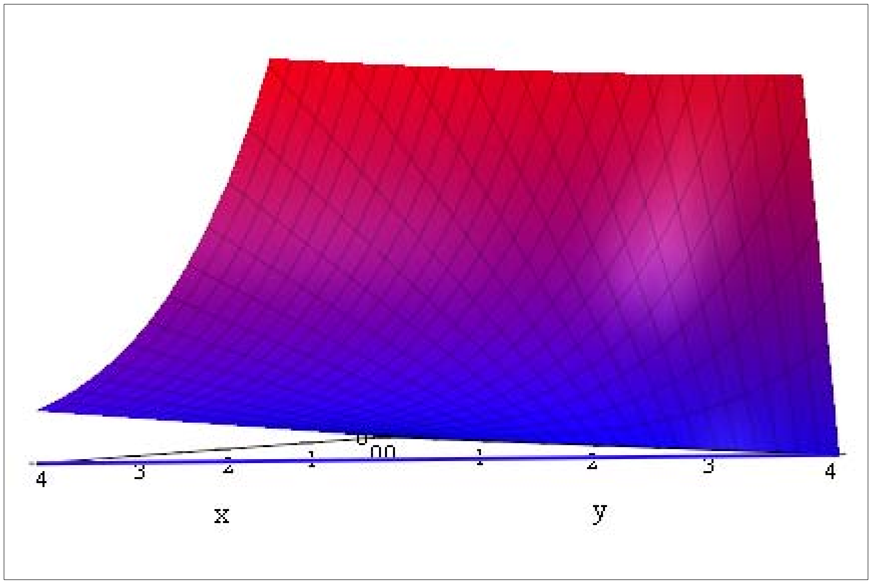
Note that

$$\int xe^{4-x}dx$$
$$=\int xe^4e^{-x}dx$$
$$=e^4 \int xe^{-x}dx$$
$$=e^4(-xe^{-x} - e^{-x})$$

Remember the method of integration by parts

$$\int_0^4 xe^{4-x}dx - \int_0^4 xdx$$
$$=e^4(-xe^{-x} - e^{-x})\Big|_0^4 - \frac{x^2}{2}\Big|_0^4$$
$$=e^4(-4e^{-4} - e^{-4} - (0 - e^0)) - \frac{4^2}{2}$$
$$=e^4(-5e^{-4} + 1) - 8$$
$$=-5 + e^4 - 8$$
$$=e^4 - 13$$

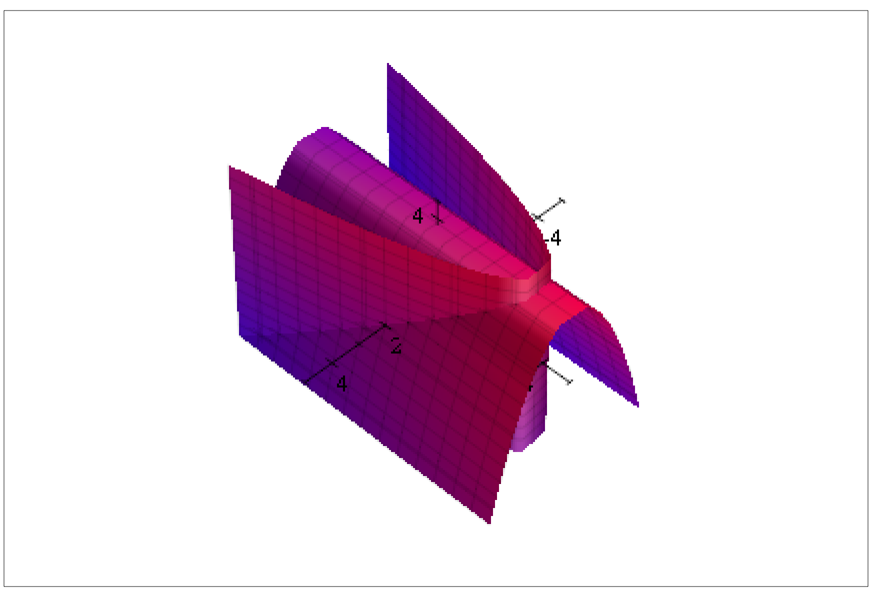
Which actually is the volume of the region under the surface and over the triangle shown below



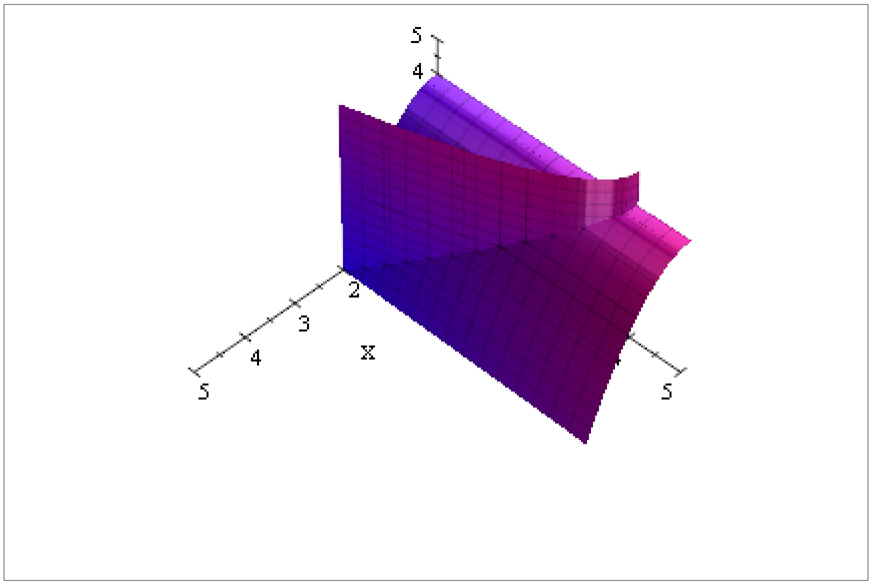
#38

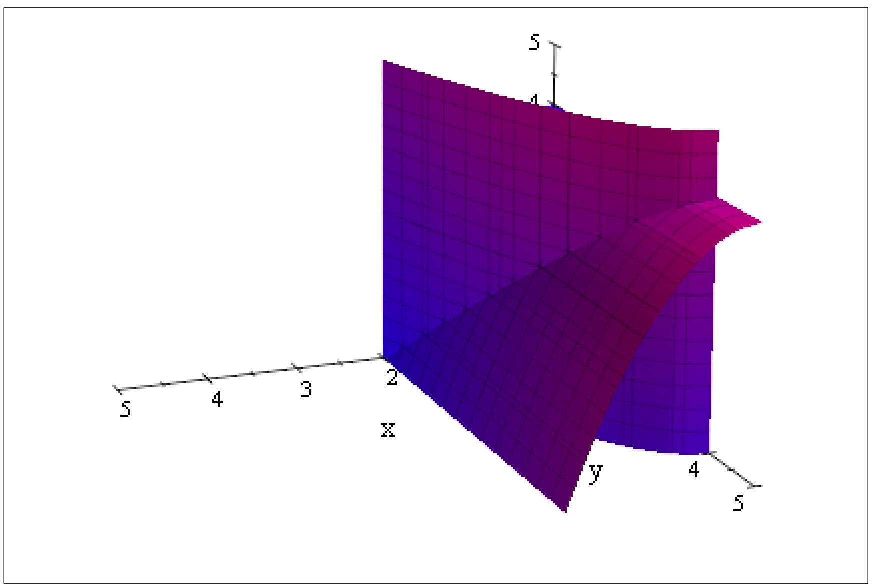
To find the volume of the solid bounded by the equations $z = 4 - x^2$ and $y = 4 - x^2$ and the first octant.

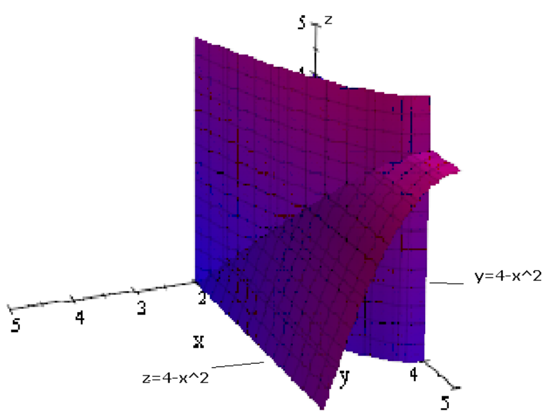
Recall the both the surfaces are cylinder with prabolas as generating curves as shown below



restricted to the first octant we have

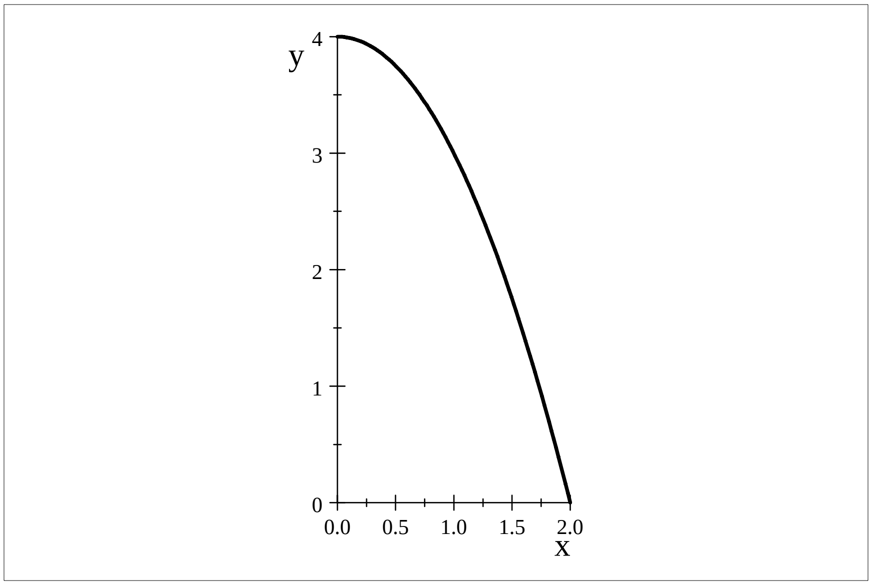


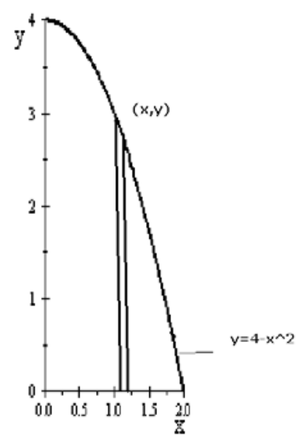




Note that the region that we are looking at is the region under the graph of $z = 4 - x^2$

and over the region enclosed by the x-axis, y-axis and graph of $y = 4 - x^2$ in the xy-plane





Therefore the desired volume is

$$\int_0^2 \int_0^{4-x^2} (4 - x^2)\,dy\,dx$$

$$= \int_0^2 \left(\int_0^{4-x^2} (4 - x^2)\,dy \right) dx$$

$$=\int_0^2 \left(4y - x^2 y\Big|_0^{4-x^2}\right)dx$$

$$=\int_0^2 (4(4-x^2) - x^2(4-x^2))dx$$

$$=\int_0^2 (4(4-x^2) - x^2(4-x^2))dx$$

$$=\int_0^2 (16 - 4x^2 - 4x^2 + x^4)dx$$

$$=\int_0^2 (16 - 8x^2 + x^4)dx$$

$$=16x - \frac{8x^3}{3} + \frac{x^5}{5}\Big|_0^2$$

$$=16(2) - \frac{8(2)^3}{3} + \frac{(2)^5}{5}$$

$$=32 - \frac{64}{3} + \frac{32}{5}$$

$$= \frac{256}{15}$$

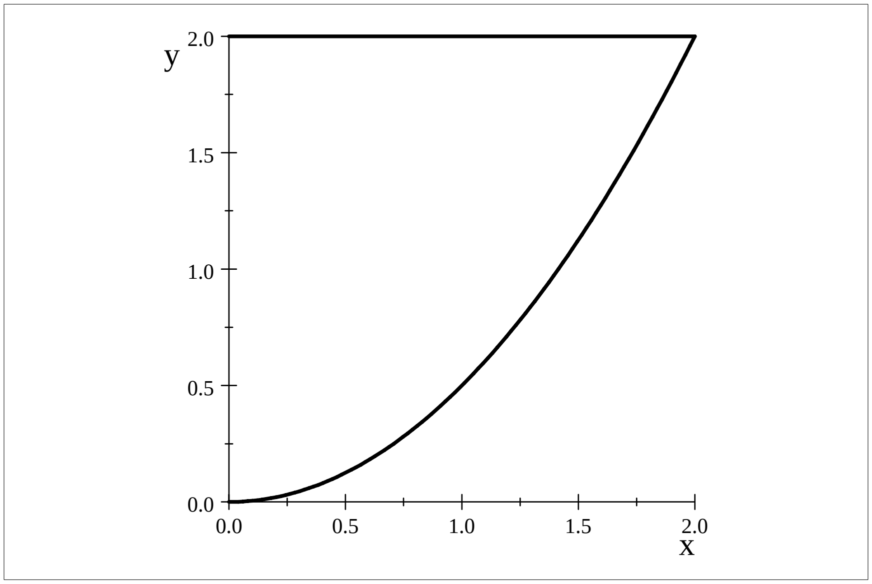
#52

To compute $\int_0^2 \int_{\frac{1}{2}x^2}^{2} \sqrt{y}\cos y\,dy\,dx$

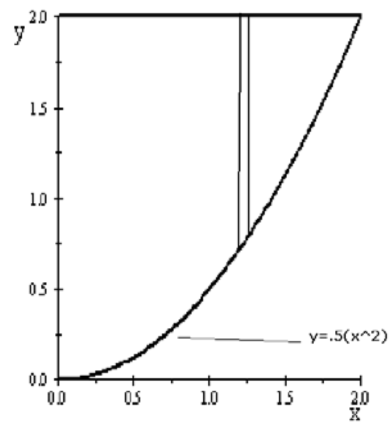
Note that it will be easier to integrate first wrt x

Note that the region R in this case is

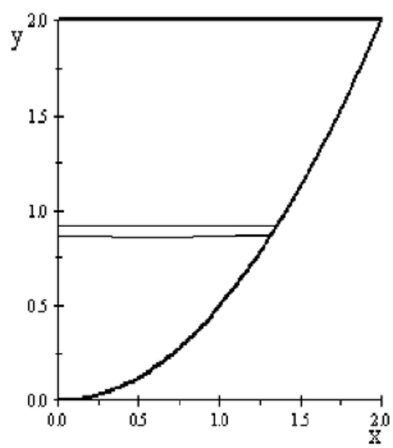
$\frac{x^2}{2}$

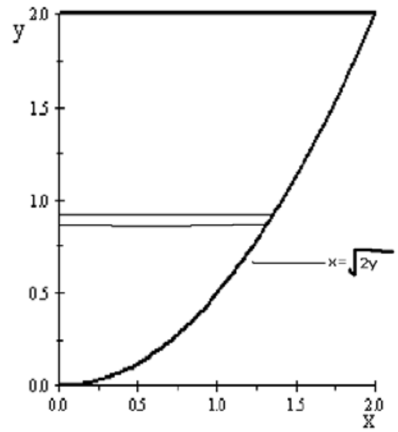


The integral is set up according to



We may change the order of the integration in the following manner





$$\int_0^2 \int_0^{\sqrt{2y}} \sqrt{y}\cos y\,dx\,dy$$
$$=\int_0^2\left(\int_0^{\sqrt{2y}} \sqrt{y}\cos y\,dx\right)dy$$
$$=\int_0^2\left(x\sqrt{y}\cos y\Big|_0^{\sqrt{2y}}\right)dy$$
$$=\int_0^2\left(\left(\sqrt{2y}\right)\sqrt{y}\cos y\right)dy$$
$$=\sqrt{2}\int_0^2 y\cos y\,dy \qquad \boxed{\text{Apply integration by parts}}$$
$$=\sqrt{2}\,(y\sin y+\cos y|_0^2)$$
$$=\sqrt{2}\,(2\sin 2+\cos 2-\cos 0)$$
$$=\sqrt{2}\,(2\sin 2+\cos 2-1)$$

#66

To show that

$$f(x,y) = \begin{cases} e^{-x-y}, & x \ge 0, y \ge 0 \\ 0, & \text{elsewhere} \end{cases}$$

is a probability density function and to find the probability

$P(0 \leq x \leq 1, x \leq y \leq 1)$

First note that $f(x,y) \geq 0$ because the non zero part is defined by an exponential function

Now the only thing that we have to show to show that f is a density function is to show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dA = 1$$

Since $f(x,y)$=0 for all the values other than $x \geq 0, y \geq 0$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dydx$$
$$= \int_{0}^{\infty} \int_{0}^{\infty} f(x,y)dydx$$
$$= \int_{0}^{\infty} \int_{0}^{\infty} e^{-x-y}dydx$$
$$= \int_{0}^{\infty} \left(Lim_{\alpha \to \infty} \int_{0}^{\alpha} e^{-x-y}dy \right) dx$$
$$= \int_{0}^{\infty} e^{-x} \left(Lim_{\alpha \to \infty} \int_{0}^{\alpha} e^{-y}dy \right) dx$$
$$= \int_{0}^{\infty} e^{-x} (Lim_{\alpha \to \infty} (-e^{-y}|_{0}^{\alpha}))dx$$
$$= \int_{0}^{\infty} e^{-x} (Lim_{\alpha \to \infty} (-e^{-\alpha} + 1))dx$$

$$=\int_0^\infty e^{-x}(1)dx$$
$$=\int_0^\infty e^{-x}dx$$
$$=1$$

and

$$P(0 \le x \le 1, x \le y \le 1) = \int_0^1 \int_x^1 e^{-x-y}dydx$$

$$\int_0^1 \int_x^1 e^{-x-y}dydx$$
$$=\int_0^1 e^{-x}\left(\int_x^1 e^{-y}dy\right)dx$$
$$=\int_0^1 e^{-x}(-e^{-y}|_x^1)dx$$
$$=\int_0^1 e^{-x}(-e^{-1}+e^{-x})dx$$
$$=-e^{-1}\int_0^1 e^{-x}dx+\int_0^1 e^{-2x}dx$$
$$=-e^{-1}(-e^{-x}|_0^1)+\left(-\frac{e^{-2x}}{2}\Big|_0^1\right)$$
$$=-e^{-1}(1-e^{-1})+\left(-\frac{e^{-2}}{2}+\frac{1}{2}\right)$$
$$=-e^{-1}+e^{-2}-\frac{e^{-2}}{2}+\frac{1}{2}$$
$$=\frac{e^{-2}}{2}-e^{-1}+\frac{1}{2}$$

Suggested Practice from section 14.2

3,9,15,17,27,37,39,49,51,63,65