Evaluation of double integrals in polar coordinats

Please read the pags 1001-1003 in the text to arrive at the

Theorem 14.3

$$\iint_R f(x,y)dA = \int_{\alpha}^{\beta}\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta$$

Exercises from section 14.3

#10

To evaluate $\int_0^{\pi/4}\int_0^4 r^2\sin\theta\cos\theta drd\theta$

$$\int_0^{\pi/4}\int_0^4 r^2\sin\theta\cos\theta drd\theta$$

$$=\int_0^{\pi/4}\left(\int_0^4 r^2\sin\theta\cos\theta dr\right)d\theta$$

$$=\int_0^{\pi/4}\left(\frac{r^3}{3}\bigg|_0^4\right)\sin\theta\cos\theta d\theta$$

$$=\int_0^{\pi/4}\left(\frac{4^3}{3}\right)\sin\theta\cos\theta d\theta$$

$$=\frac{64}{3}\int_0^{\pi/4}\sin\theta\cos\theta d\theta \quad , \quad \text{set } \sin\theta = u \rightarrow \cos\theta d\theta = du$$

$$=\frac{64}{3}\int_0^{\sqrt{2}/2} udu$$

$$= \frac{64}{3}\left(\frac{u^2}{2} \Big|_0^{\sqrt{2}/2} \right)$$
$$= \frac{64}{3}\left(\frac{2}{8} \right)$$
$$= \frac{16}{3}$$

#18

To evaluate $\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\, dxdy$

by using polar coordinates

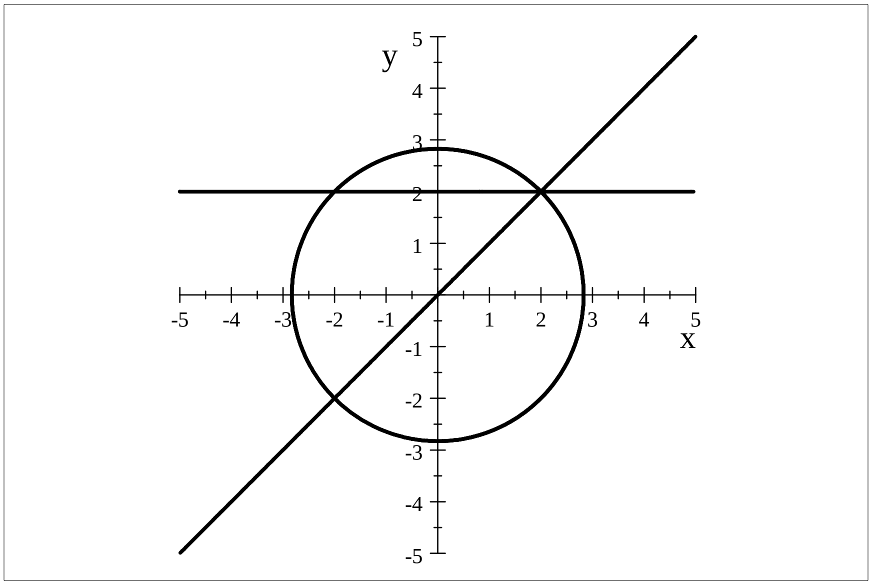
$x = r\cos\theta \qquad y = r\sin\theta$

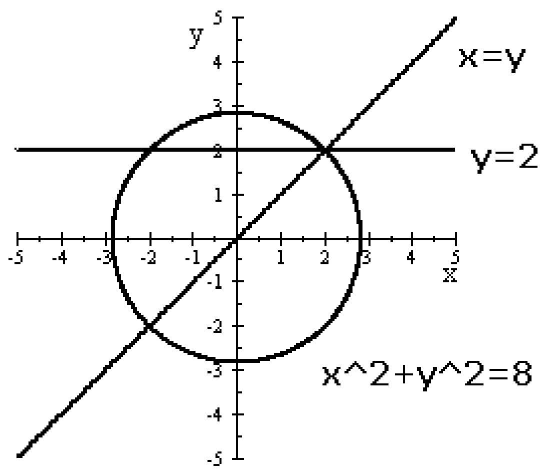
$dxdy \to rdrd\theta$

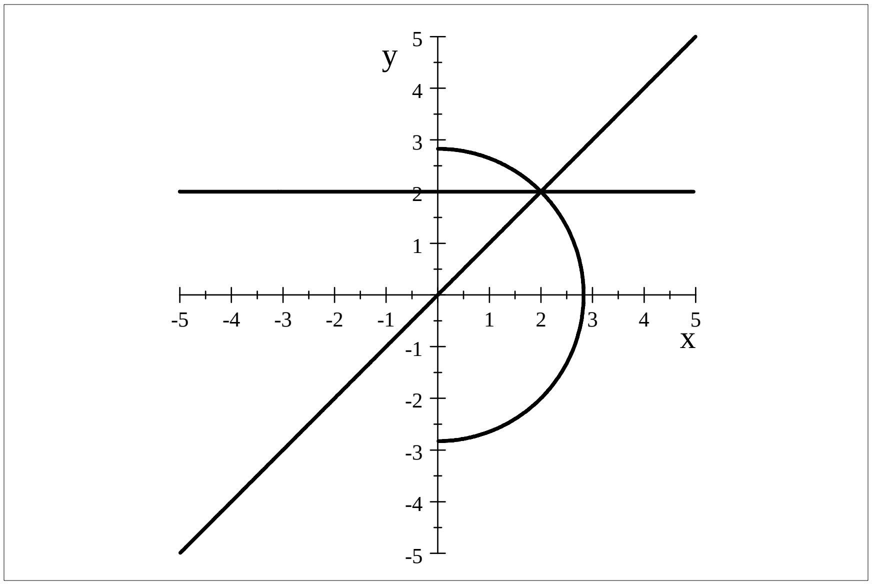
The region of integration is

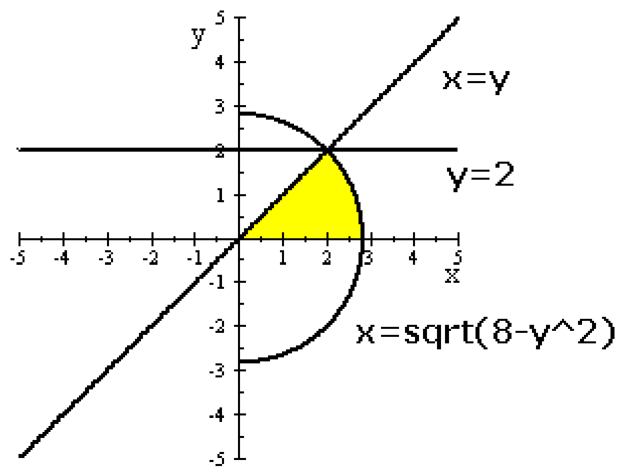
$0 \le y \le 2$, $y \le x \le \sqrt{8-y^2}$

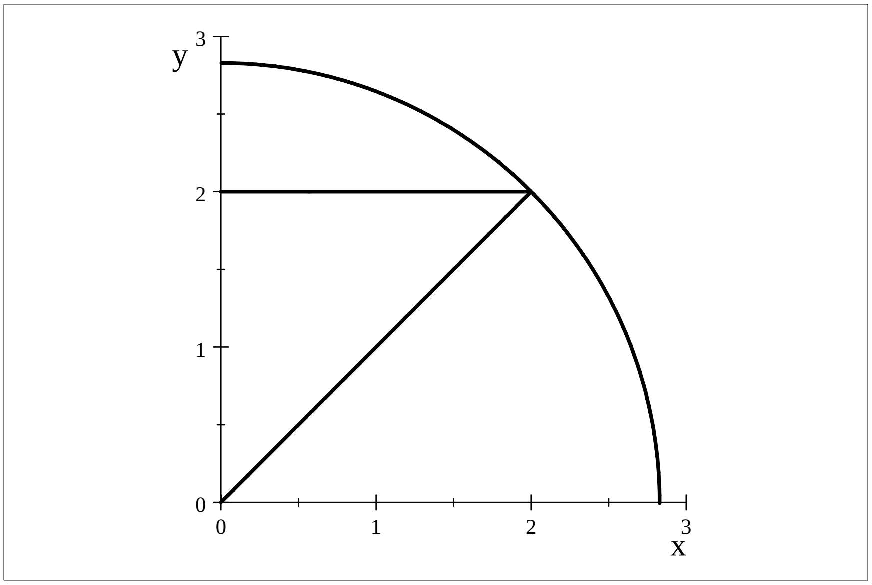
$x^2 + y^2 = 8$

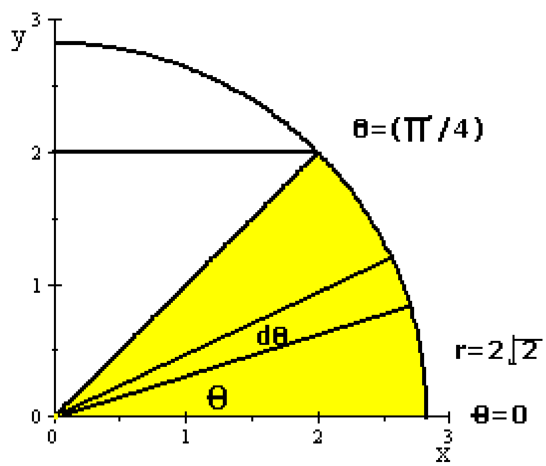

y
x


y
x
x=y
y=2
x^2+y^2=8


y
x


y
x
x=y
y=2
x=sqrt(8-y^2)





$$\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\, dxdy$$

$$= \int_0^{\pi/4} \int_0^{2\sqrt{2}} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\, rdrd\theta$$

$$= \int_0^{\pi/4} \int_0^{2\sqrt{2}} \sqrt{r^2(\cos^2\theta + \sin^2\theta)}\, rdrd\theta$$

$$= \int_0^{\pi/4} \int_0^{2\sqrt{2}} \sqrt{r^2}\, rdrd\theta$$

$$= \int_0^{\pi/4} \int_0^{2\sqrt{2}} r r dr d\theta \qquad r \geq 0$$

$$= \int_0^{\pi/4} \int_0^{2\sqrt{2}} r^2 dr d\theta$$

$$= \int_0^{\pi/4} \left(\int_0^{2\sqrt{2}} r^2 dr \right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{r^3}{3} \Big|_0^{2\sqrt{2}} \right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{8 \times 2\sqrt{2}}{3} \right) d\theta$$

$$= \frac{16\sqrt{2}}{3} \int_0^{\pi/4} d\theta$$

$$= \frac{16\sqrt{2}}{3} \left(\frac{\pi}{4} \right)$$

$$= \frac{4\sqrt{2}\,\pi}{3}$$

#20
To evaluate

$$\int_0^4 \int_0^{\sqrt{4y-y^2}} x^2 dx dy$$

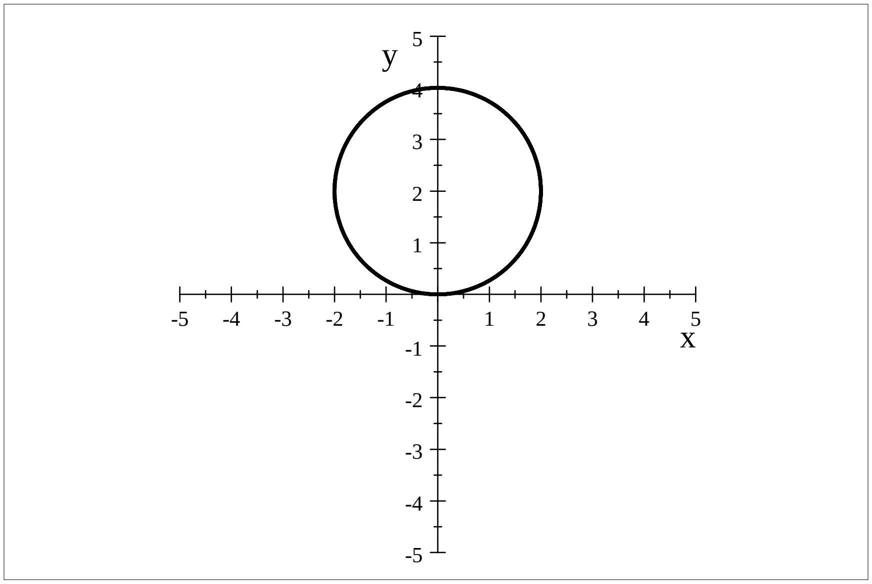
by using polar coordinates

First, let us look at that the region of integration

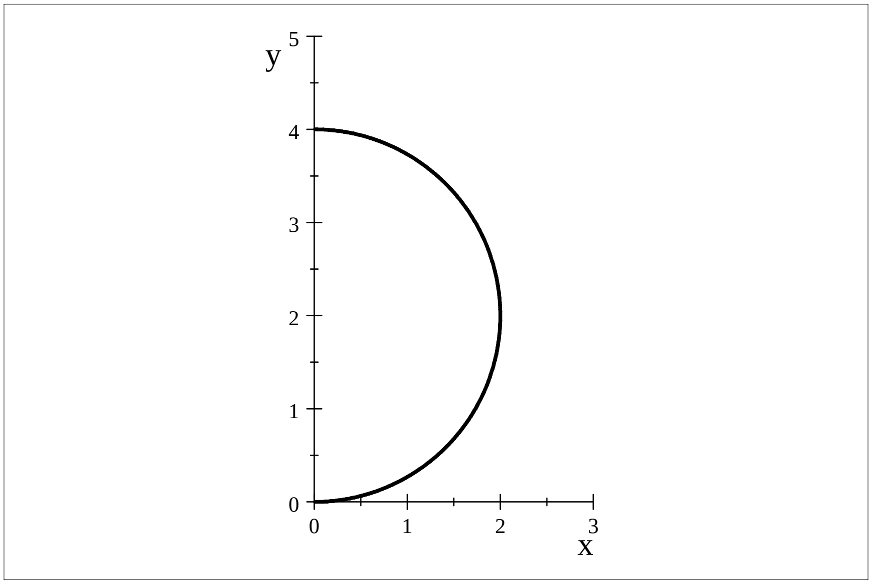
$0 \leq y \leq 4 \qquad 0 \leq x \leq \sqrt{4y - y^2}$

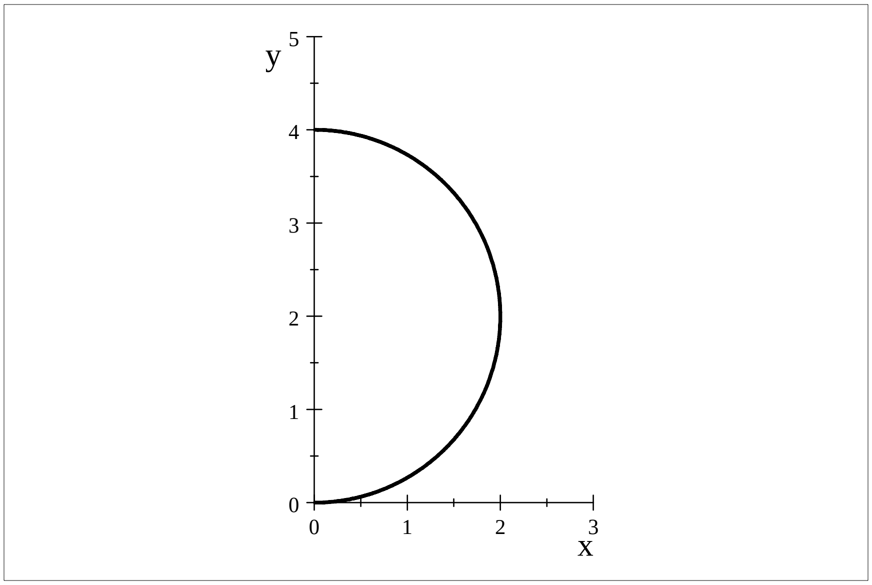
note that $x = \sqrt{4y - y^2}$ is right half of the circle

$x^2 = 4y - y^2$
$x^2 + y^2 - 4y = 0$
$x^2 + y^2 - 4y + 4 = 4$
$x^2 + (y-2)^2 = 4$

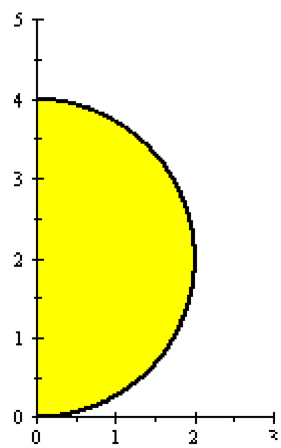


Recall that the polar equation of the above circle is $r = 4\sin\theta$
The region of integration is

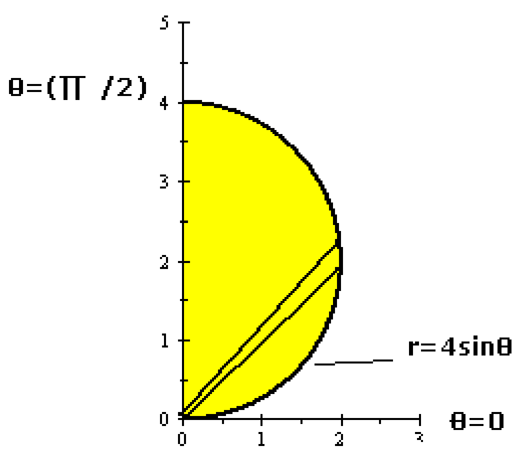




y
5
4
3
2
1
0
0
1
2
3
x



5
4
3
2
1
0
0
1
2



θ=(Π /2)
5
4
3
2
1
0
0
1
2
r=4sinθ
θ=0

$$\int_0^4 \int_0^{\sqrt{4y-y^2}} x^2 dxdy$$

$$= \int_0^{\pi/2} \int_0^{4\sin\theta} r^2 \cos^2\theta r dr d\theta$$

$$= \int_0^{\pi/2} \int_0^{4\sin\theta} r^3 \cos^2\theta dr d\theta$$

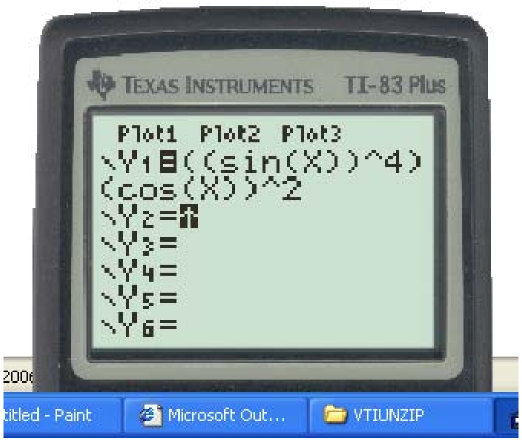
$$= \int_0^{\pi/2} \left(\int_0^{4\sin\theta} r^3 \cos^2\theta dr \right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^4 \cos^2\theta}{4} \Big|_0^{4\sin\theta} \right) d\theta$$

$$= \int_0^{\pi/2} (64 \sin^4\theta \cos^2\theta) d\theta$$

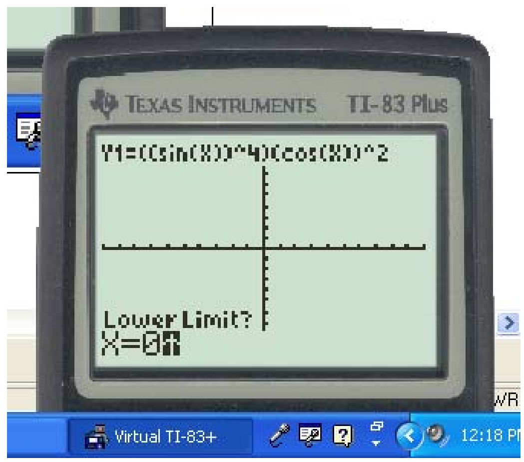
$$=64 \int_0^{\pi/2} \sin^4\theta \cos^2\theta d\theta$$

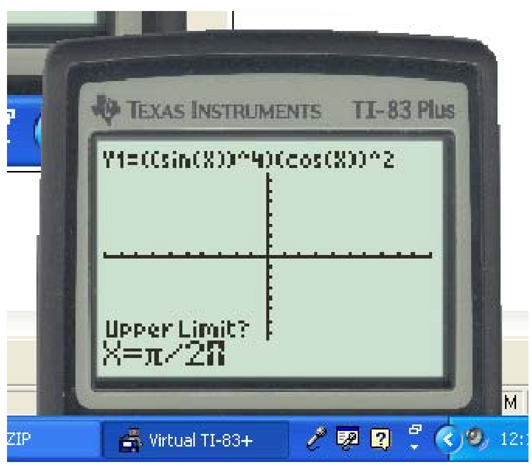
$= 2\pi$ (The details of the steps are shown in the footnote. On the Final Exam, unless I ask you for the exact value of the answer, you may give a calculator approximation as shown below.)

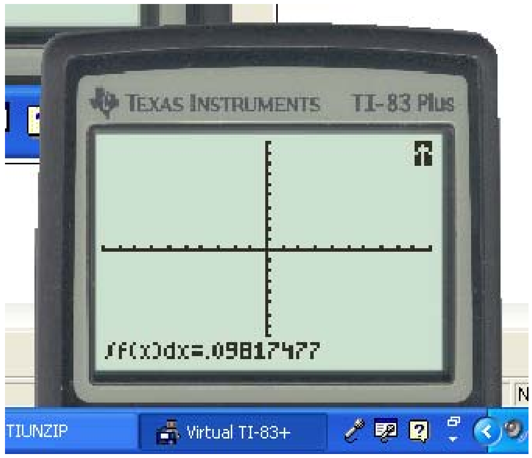


Choose calc
and then $\int f(x)dx$


TEXAS INSTRUMENTS TI-83 Plus
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
VTIUNZIP
Virtual TI-83+


TEXAS INSTRUMENTS TI-83 Plus
Y1=((sin(X))^4)(cos(X))^2
Lower Limit?
X=0
Virtual TI-83+
12:18 PM


TEXAS INSTRUMENTS TI-83 Plus
Y1=((sin(X))^4)(cos(X))^2
Upper Limit?
X=π/2
Virtual TI-83+



$64 \times .09817477 = 6.28318528$

#22 This is a very cute example to show how the polar transformation will combine a two step integration process (of the same function) into one integral.

To write

$\int_0^{5\sqrt{2}/2} \int_0^{x} xydydx + \int_{5\sqrt{2}/2}^{5} \int_0^{\sqrt{25-x^2}} xydydx$ as a single integral and find the value

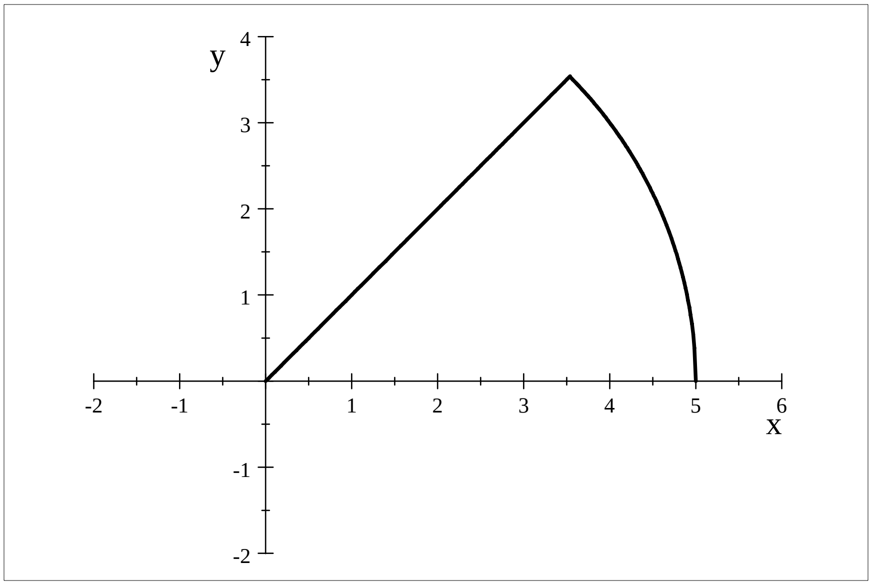
For the first term

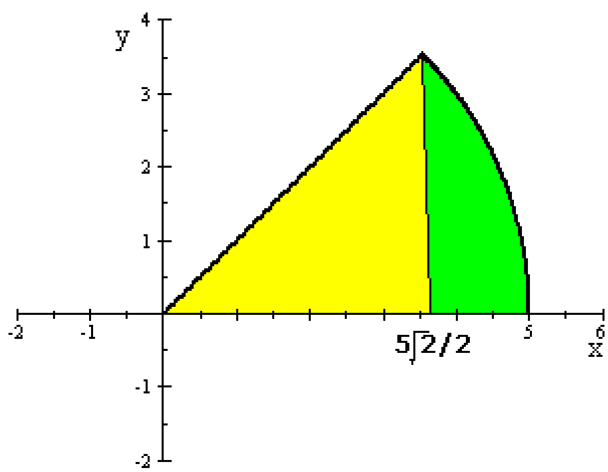
$0 \le x \le \frac{5\sqrt{2}}{2}$, $0 \le y \le x$ (YELLOW REGION)

the second term

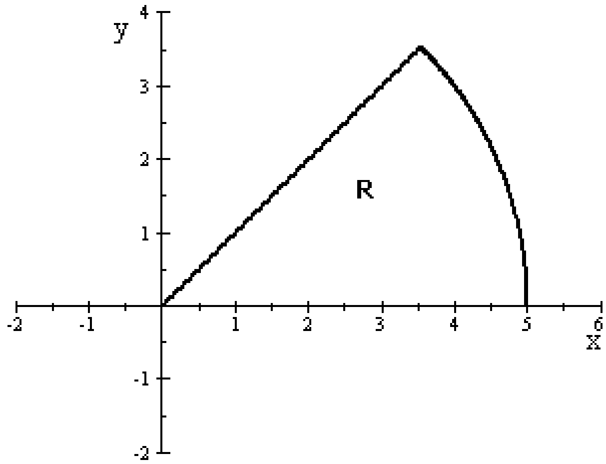
$\frac{5\sqrt{2}}{2} \le x \le 5$, $0 \le y \le \sqrt{25 - x^2}$ (green region)

x





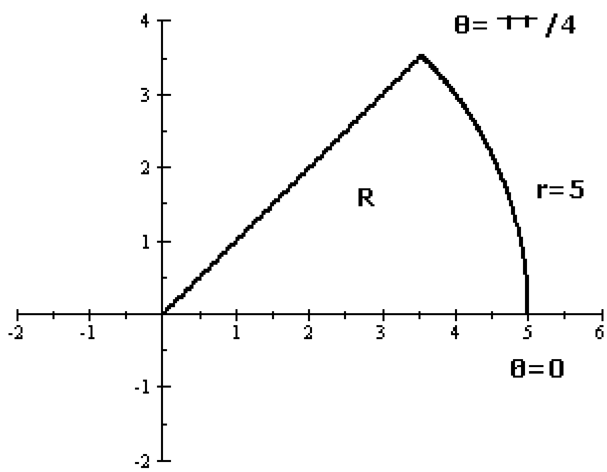
We can combine these into one region *R*



$$\int_0^{5\sqrt{2}/2}\int_0^{x} xy\,dy\,dx + \int_{5\sqrt{2}/2}^{5}\int_0^{\sqrt{25-x^2}} xy\,dy\,dx$$

$$=\iint_R xy\,dA$$

transform to polars



$$\iint_R xy\,dA$$

$$= \int_0^{\pi/4}\int_0^{5}(r\cos\theta)(r\sin\theta)r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^5 r^3 \cos\theta \sin\theta dr d\theta$$

$$= \int_0^{\pi/4} \left(\int_0^5 r^3 \cos\theta \sin\theta dr \right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{r^4 \cos\theta \sin\theta}{4} \Big|_0^5 \right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{5^4 \cos\theta \sin\theta}{4} \right) d\theta$$

$$= \frac{625}{4} \int_0^{\pi/4} \sin\theta \cos\theta d\theta$$

$$= \frac{625}{4} \int_0^{\sqrt{2}/2} u du \qquad \boxed{u = \sin\theta}$$

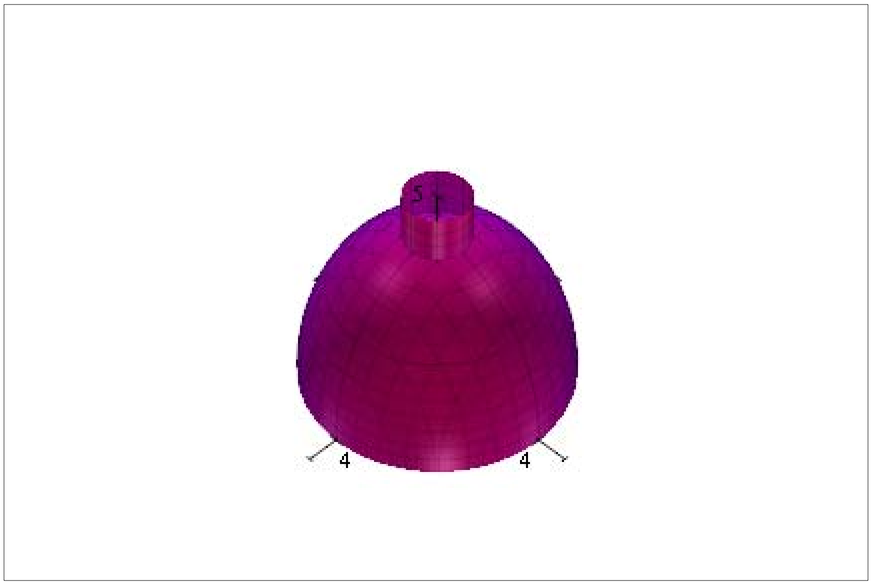
$$= \frac{625}{4} \left(\frac{u^2}{2} \Big|_0^{\sqrt{2}/2} \right)$$

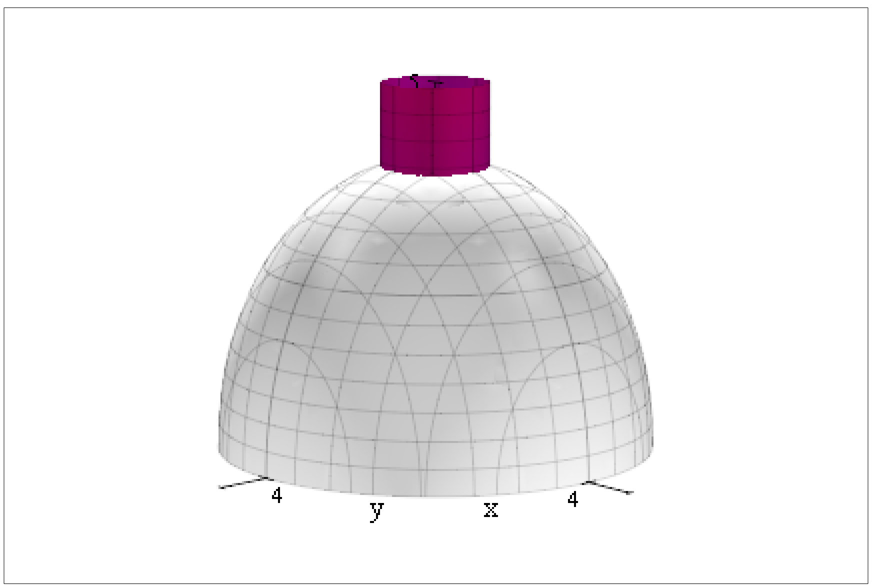
$$= \frac{625}{16}$$

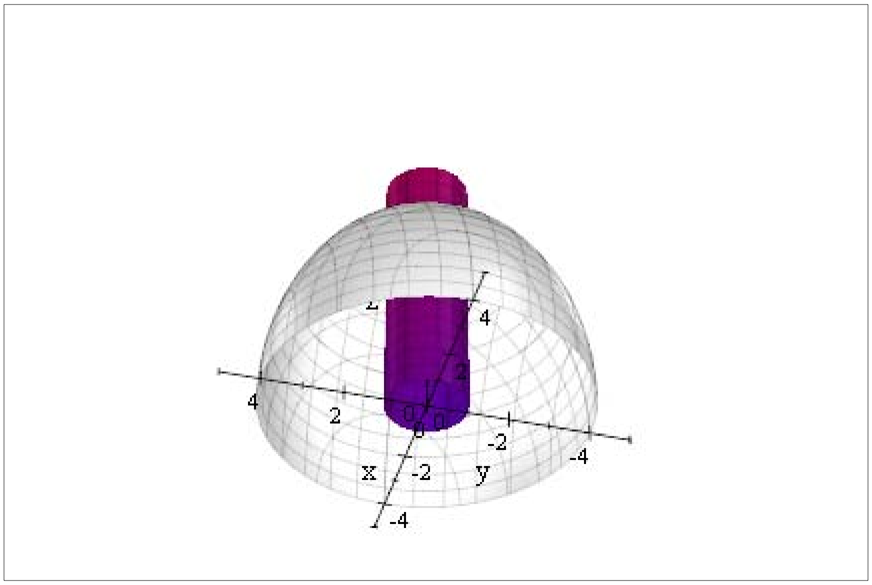
#32

To find find the volume of the solid that is inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = 1$

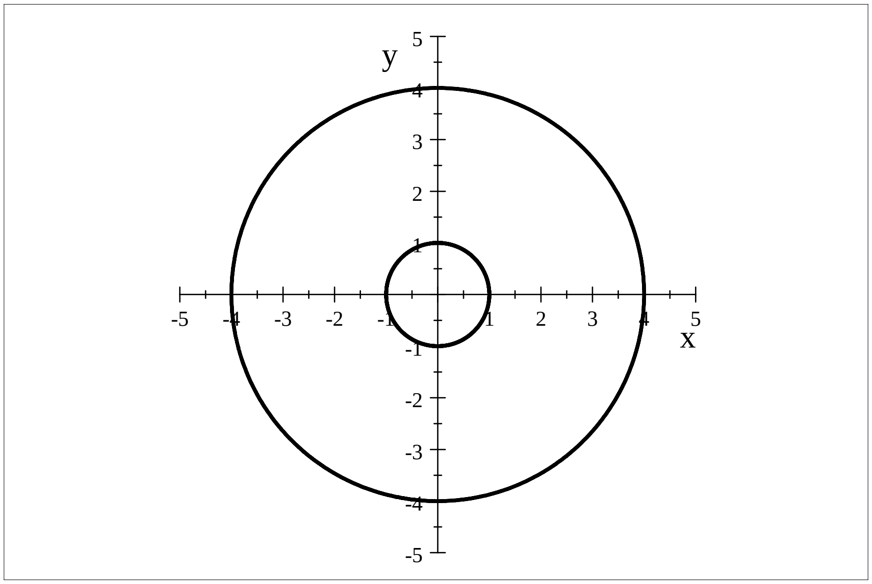
Let us look at a set of graphs of these surfaces

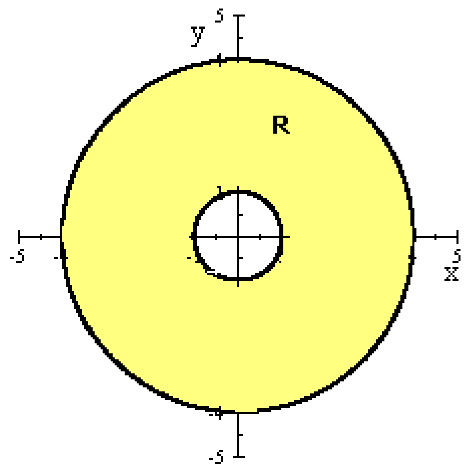

5
4
4


4
y
x
4



The desired valume is the volume under the graph of the region below the surface $z = \sqrt{16 - x^2 - y^2}$ over the region R shown below





$$\iint_R \sqrt{16 - x^2 - y^2}\, dA$$

changing to polars

$$\iint_R \sqrt{16 - x^2 - y^2}\, dA$$

$$= \int_0^{2\pi} \int_1^4 \sqrt{16 - r^2}\, r dr d\theta$$

$$= \int_0^{2\pi} \left(\int_1^4 \sqrt{16 - r^2}\, r dr \right) d\theta$$

$$= \int_0^{2\pi} \left(\int_{15}^0 \left(-\frac{1}{2} \right) \sqrt{u}\, du \right) d\theta \qquad \boxed{16 - r^2 = u}$$

$$= \int_0^{2\pi} \left[-\left(\frac{1}{2} \right) \left(\frac{2}{3} u^{3/2} \right) \Big|_{15}^{0} \right] d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} u^{3/2} \Big|_{15}^{0} \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{3} u^{3/2} \Big|_{0}^{15} \right] d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} (15)^{3/2} d\theta$$

$$=\frac{1}{3}\int_0^{2\pi} 15^{3/2}d\theta$$
$$=\frac{15^{3/2}}{3}\int_0^{2\pi} d\theta$$
$$=\frac{15^{3/2}(2\pi)}{3}$$
$$=\frac{15\sqrt{15}\,(2\pi)}{3}$$
$$=10\sqrt{15}\,\pi$$

Suggested Practice in section 14.3

7,15,17,19,21,29,31,33,55

Footnote

Evaluation of the integral involved in the exercise #20

$\int \sin^4\theta\cos^2\theta d\theta$, we just have to play with this integral in several different ways

$$\int \sin^4\theta(1-\sin^2\theta)d\theta$$
$$=\int(\sin^4\theta-\sin^6\theta)d\theta$$

We may try the reduction of powers techniques from the integration by parts

$$\int \sin^6\theta d\theta$$
$$=\int \sin^5\theta\sin\theta d\theta \qquad \boxed{u=\sin^5\theta, dv=\sin\theta d\theta}$$
$$=\sin^5\theta(-\cos\theta)-\int(-\cos\theta)(5\sin^4\theta\cos\theta)d\theta$$
$$=-\sin^5\theta\cos\theta+5\int \sin^4\theta\cos^2\theta d\theta$$
$$=-\sin^5\theta\cos\theta+5\int \sin^4\theta(1-\sin^2\theta)d\theta$$
$$=-\sin^5\theta\cos\theta+5\int(\sin^4\theta-\sin^6\theta)d\theta$$

$$=-\sin^5\theta\cos\theta + 5\int \sin^4\theta d\theta - 5\int \sin^6\theta d\theta$$

$$\int \sin^6\theta d\theta = -\sin^5\theta\cos\theta + 5\int \sin^4\theta d\theta - 5\int \sin^6\theta d\theta$$

that is

$$\boxed{\int \sin^6\theta d\theta} = -\sin^5\theta\cos\theta + 5\int \sin^4\theta d\theta - 5\boxed{\int \sin^6\theta d\theta}$$

$$\boxed{\int \sin^6\theta d\theta} + 5\boxed{\int \sin^6\theta d\theta} = -\sin^5\theta\cos\theta + 5\int \sin^4\theta d\theta$$

$$6\int \sin^6\theta d\theta = -\sin^5\theta\cos\theta + 5\int \sin^4\theta d\theta$$

$$\int \sin^6\theta d\theta = -\frac{1}{6}\sin^5\theta\cos\theta + \frac{5}{6}\int \sin^4\theta d\theta$$

$$\int \sin^4\theta(1-\sin^2\theta)d\theta$$
$$=\int(\sin^4\theta - \sin^6\theta)d\theta$$
$$=\int \sin^4\theta d\theta - \int \sin^6\theta d\theta$$
$$=\int \sin^4\theta d\theta - \int \sin^6\theta d\theta$$
$$=\int \sin^4\theta d\theta - \left(-\frac{1}{6}\sin^5\theta\cos\theta + \frac{5}{6}\int \sin^4\theta d\theta\right)$$
$$=\int \sin^4\theta d\theta + \frac{1}{6}\sin^5\theta\cos\theta - \frac{5}{6}\int \sin^4\theta d\theta$$
$$=\frac{1}{6}\sin^5\theta\cos\theta + \frac{1}{6}\int \sin^4\theta d\theta$$

Continue with the technique of integartion by parts

$$\int \sin^4\theta d\theta = \sin^3\theta(-\cos\theta) - \int(-\cos\theta)(3\sin^2\theta\cos\theta)d\theta$$

$$\int \sin^4\theta d\theta = \sin^3\theta(-\cos\theta) + \int 3\sin^2\theta\cos^2\theta d\theta$$

$$\int \sin^4\theta d\theta = -\sin^3\theta\cos\theta + 3\int \sin^2\theta(1-\sin^2\theta)d\theta$$

$$\int \sin^4\theta d\theta = -\sin^3\theta\cos\theta + 3\int \sin^2\theta d\theta - 3\int \sin^4\theta d\theta$$

$$\int \sin^4\theta d\theta + 3\int \sin^4\theta d\theta = -\sin^3\theta\cos\theta + 3\int \sin^2\theta d\theta$$

$$4\int \sin^4\theta d\theta = -\sin^3\theta\cos\theta + 3\int \sin^2\theta d\theta$$

$$\int \sin^4\theta d\theta = -\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\int \sin^2\theta d\theta$$

$$\int \sin^2\theta d\theta = \int\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta$$

$$\int \sin^4\theta\cos^2\theta d\theta$$

$$= \frac{1}{6}\sin^5\theta\cos\theta + \frac{1}{6}\int \sin^4\theta d\theta$$

$$= \frac{1}{6}\sin^5\theta\cos\theta + \frac{1}{6}\left(-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\int \sin^2\theta d\theta\right)$$

$$= \frac{1}{6}\left(\sin^5\theta\cos\theta - \frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\int \sin^2\theta d\theta\right)$$

$$= \frac{1}{6}\left(\sin^5\theta\cos\theta - \frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\right)$$

$$= \frac{1}{6}\left(\sin^5\theta\cos\theta - \frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{8}\theta - \frac{3}{16}\sin 2\theta\right)$$

$$\int_0^{\pi/2} \sin^4\theta\cos^2\theta d\theta$$

$$= \frac{1}{6}\left(\sin^5\theta\cos\theta - \frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{8}\theta - \frac{3}{16}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{6}\left(\frac{3}{8}\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{\pi}{32}$$

$$64\int_0^{\pi/2} \sin^4\theta\cos^2\theta d\theta = 64\left(\frac{\pi}{32}\right) = 2\pi$$