Read the pages 982-987 on iterated integrals and their applications to compute the areas.

I shall demonstrate you the procedure by using some exercises from section 14.1 (pages 988-990)

#14

To evaluate the iterated integral

$$\int_0^4 \int_1^{\sqrt{x}} 2ye^{-x} dy dx$$

$$= \int_0^4 \left(\int_1^{\sqrt{x}} 2ye^{-x} dy \right) dx$$, we shall integrate the inside integral wrt y treating x as a constant

$$= \int_1^4 \left(\int_1^{\sqrt{x}} 2ye^{-x} dy \right) dx$$

$$= \int_1^4 \left(2\left(\frac{y^2}{2}\right) e^{-x} \Big|_1^{\sqrt{x}} \right) dx$$

$$= \int_1^4 \left(y^2 e^{-x} |_1^{\sqrt{x}} \right) dx$$

$$= \int_1^4 (xe^{-x} - e^{-x}) dx$$

$$= \int_1^4 (xe^{-x} - e^{-x}) dx$$

Use the integration by parts to evaluate $\int xe^{-x} dx$

A review of integration by parts is posted as a separate file

Remember that

$$\int udv = uv - \int vdu$$

$\int xe^{-x}dx$

take $u = x \to du = dx$

$dv = e^{-x}dx \to v = \int e^{-x}dx \to v = -e^{-x}$

$\int xe^{-x}dx = x(-e^{-x}) - \int(-e^{-x})dx$

$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx$

$\int(xe^{-x} - e^{-x})dx = \int xe^{-x}dx - \int e^{-x}dx = -xe^{-x} + \int e^{-x}dx - \int e^{-x}dx = -xe^{-x}$

$$\int_1^4(xe^{-x} - e^{-x})dx = -xe^{-x}|_1^4 = -4e^{-4} + e^{-1} = \frac{1}{e} - \frac{4}{e^4}$$

#24

$$\int_0^{\pi/4}\int_0^{\cos\theta} 3r^2\sin\theta dr d\theta$$

$$= \int_0^{\pi/4}\left(\int_0^{\cos\theta} 3r^2\sin\theta dr\right)d\theta$$

$$= \int_0^{\pi/4}\left(r^3\sin\theta|_0^{\cos\theta}\right)d\theta$$

$$= \int_0^{\pi/4}(\cos^3\theta\sin\theta - 0)d\theta$$

$$= \int_0^{\pi/4}\cos^3\theta\sin\theta d\theta$$

$$\boxed{\cos\theta = u \to -\sin\theta d\theta = du;\ \ \theta = 0 \to \cos\theta = 1;\ \theta = \frac{\pi}{4} \to \cos\theta = \frac{\sqrt{2}}{2}}$$

$$\int_0^{\pi/4}\int_0^{\cos\theta} 3r^2\sin\theta dr d\theta$$

$$=-\int_{1}^{\sqrt{2}/2} u^3 du$$
$$=-\frac{u^4}{4}\Big|_{1}^{\sqrt{2}/2}$$
$$=-\frac{\left(\sqrt{2}/2\right)^4}{4}+\frac{1}{4}$$
$$=\frac{1}{4}-\frac{\left(\sqrt{2}/2\right)^4}{4}$$
$$=\frac{1}{4}-\frac{1}{16}$$
$$=\frac{3}{16}$$

#26

To evaluate $\int_0^3\int_0^\infty \frac{x^2}{1+y^2}dydx$

$$\int_0^\infty \frac{x^2}{1+y^2}dy$$
$=Lim_{\alpha\to\infty}\int_0^\alpha \frac{x^2}{1+y^2}dy$, treat x as a constant
$$=Lim_{\alpha\to\infty}(x^2\tan^{-1}y|_0^\alpha)$$
$$=Lim_{\alpha\to\infty}(x^2\tan^{-1}\alpha-0)$$
$$=x^2\left(\frac{\pi}{2}\right)$$
$$=\frac{\pi x^2}{2}$$

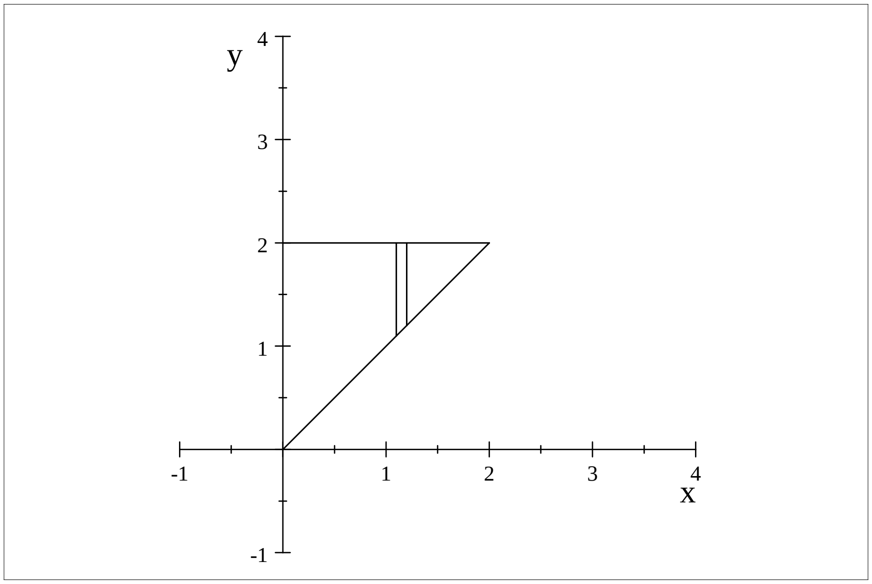
$$\int_0^3\int_0^\infty \frac{x^2}{1+y^2}dydx$$
$$=\int_0^3 \frac{\pi x^2}{2}dx$$
$$=\frac{\pi}{2}\left(\frac{x^3}{3}\Big|_0^3\right)$$

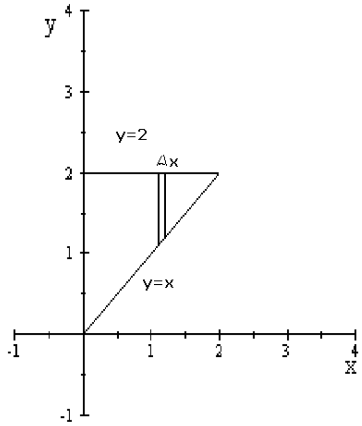
$$=\frac{\pi}{2}\left(\frac{3^3}{3}\right)$$
$$=\frac{9\pi}{2}$$

#62

$$\int_0^2\int_x^2 e^{-y^2}dydx$$

Note that we are integrating over the following region



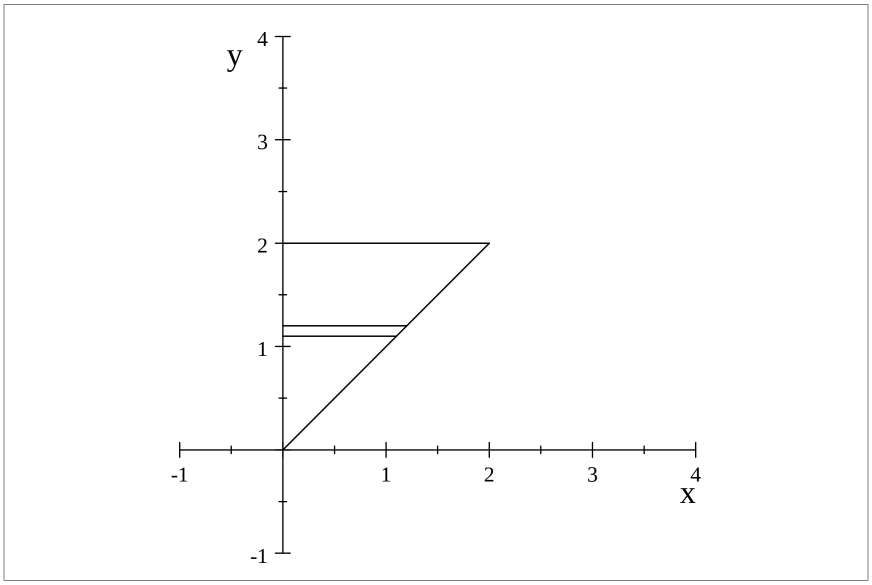


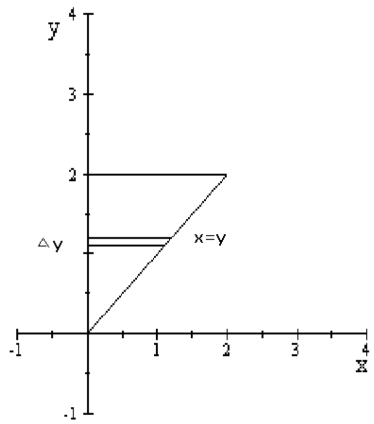
The problem in keeping the present order of integration for the evaluation of

$$\int_0^2 \int_x^2 e^{-y^2} dy dx$$

is that for $\int_x^2 e^{-y^2} dy$ we do not know the antiderivative of e^{-y^2}

Therefore we may consider changing the order of integration to integrating first wrt x first and y next





$\int_0^2 \int_x^2 e^{-y^2} dy dx$ may be written as $\int_0^2 \int_0^y e^{-y^2} dx dy$

Then

$$\int_0^2 \int_0^y e^{-y^2} dx dy$$
$$= \int_0^2 \left(\int_0^y e^{-y^2} dx \right) dy \quad \text{(treat } y \text{ therefore } e^{-y^2} \text{ as a constant here)}$$
$$= \int_0^2 \left(x e^{-y^2} \Big|_0^y \right) dy$$
$$= \int_0^2 \left(y e^{-y^2} - 0 \right) dy$$
$$= \int_0^2 y e^{-y^2} dy$$
$$= \int_0^{-4} -\frac{e^u}{2} du$$

sub $-y^2 = u \rightarrow -2y dy = du$ and change the limits of integration accordingly

$$= -\frac{e^u}{2} \Big|_0^{-4}$$
$$= -\frac{e^{-4}}{2} + \frac{1}{2}$$
$$= \frac{1}{2}(1 - e^{-4})$$

Suggested Practice for Section 14.1

9,11,13,15,21,25,27,61,63