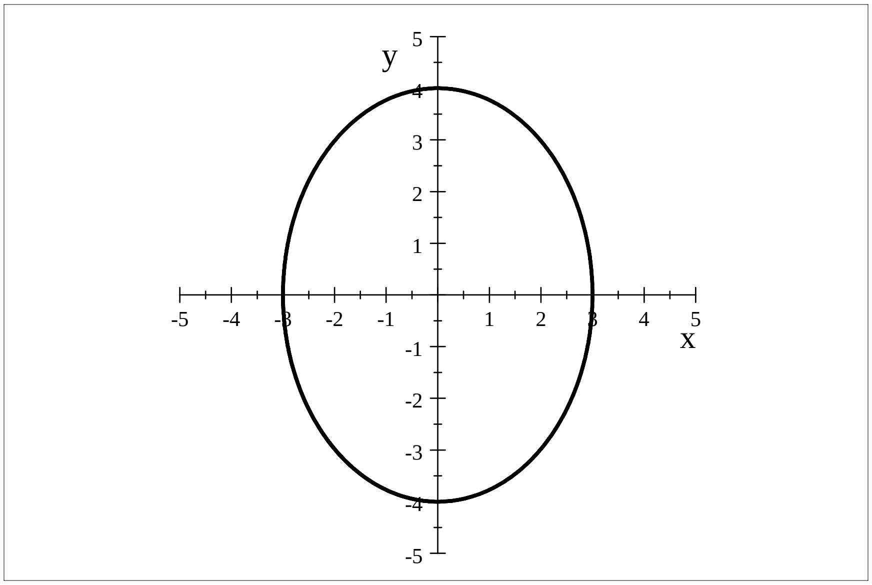
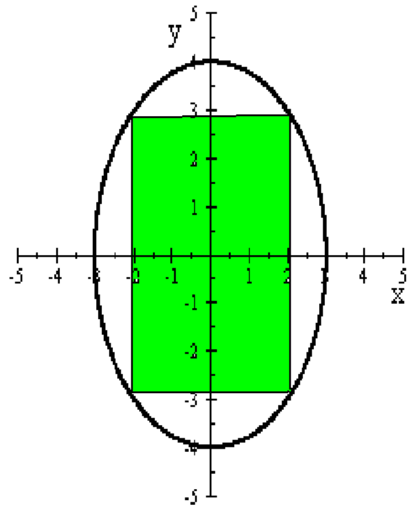
We are going to work on the method of Lagrange Multiplier to find the maximum or minimum value of a function subject to a constraint.

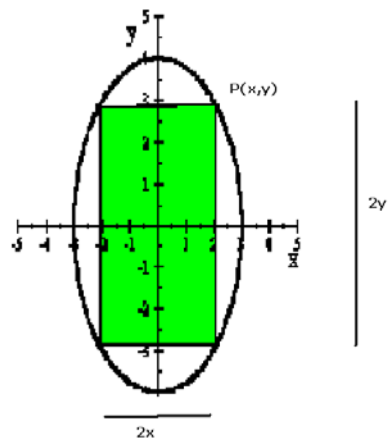
Example:

We would like to construct a rectangle of largest possible area that can be inscribed in the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$



we would like to obtain the dimensions x and y such that the area of the inscribed rectangle (like the green rectangle shown below) is maximum.

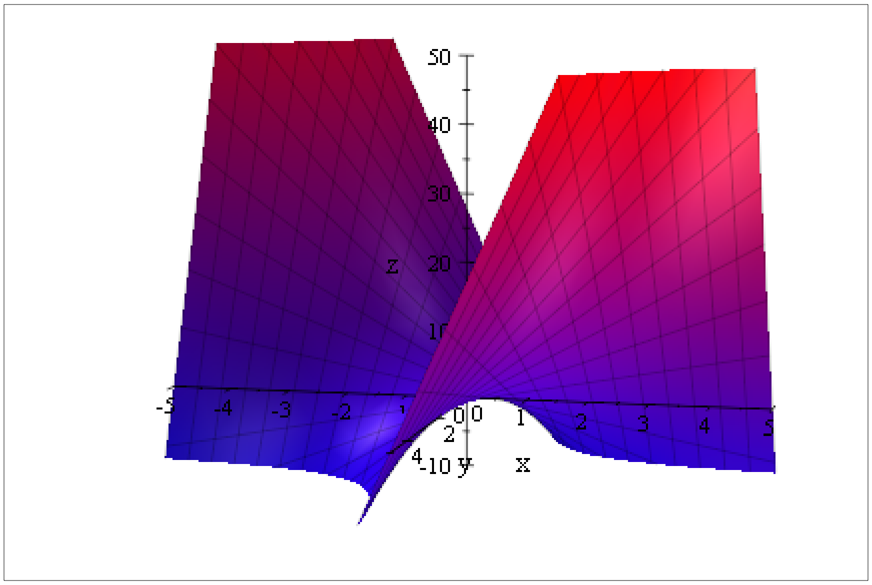




The green area is $4xy$

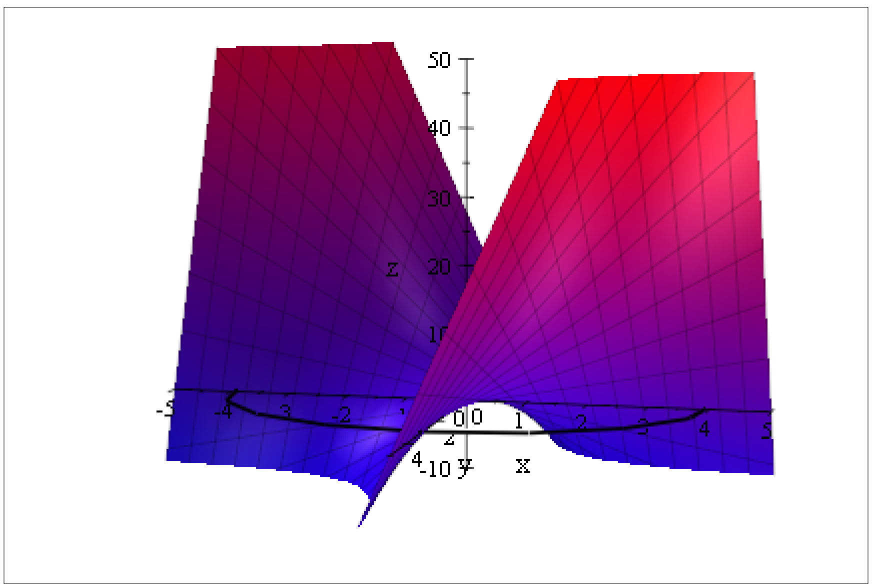
In other words, we would like to maximize the function $f(x,y) = 4xy$ subject to the constraint $\frac{x^2}{9} + \frac{y^2}{16} = 1$

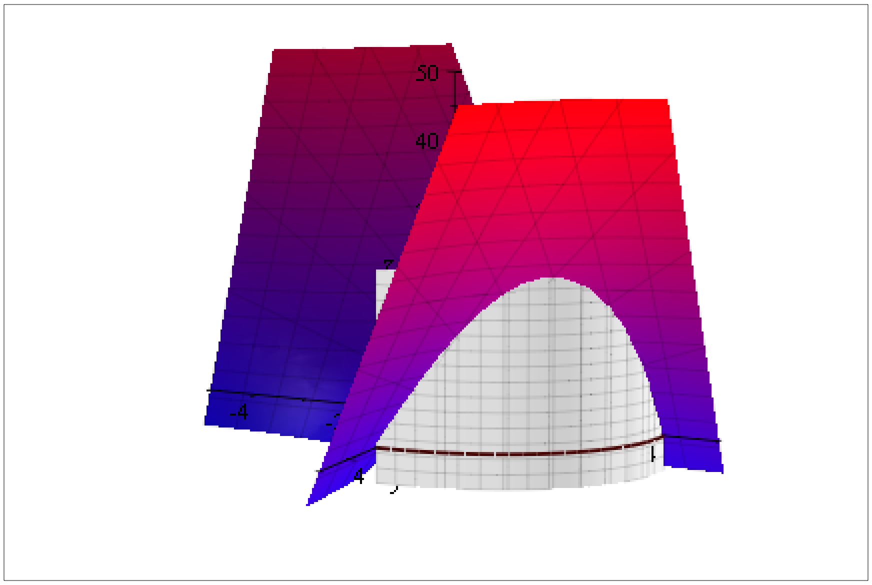
A graph of the function (of two variables) is shown below.



We would like the find the heighest z-value on the graph of $f(x,y)$

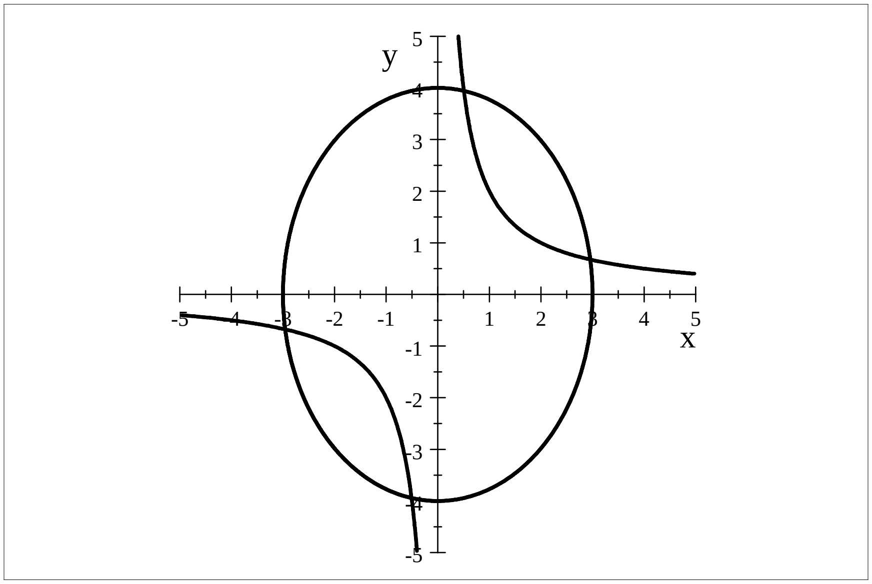
when looking directly above from the graph of the ellipse
$\frac{x^2}{9} + \frac{y^2}{16} = 1$ (shown in black color)



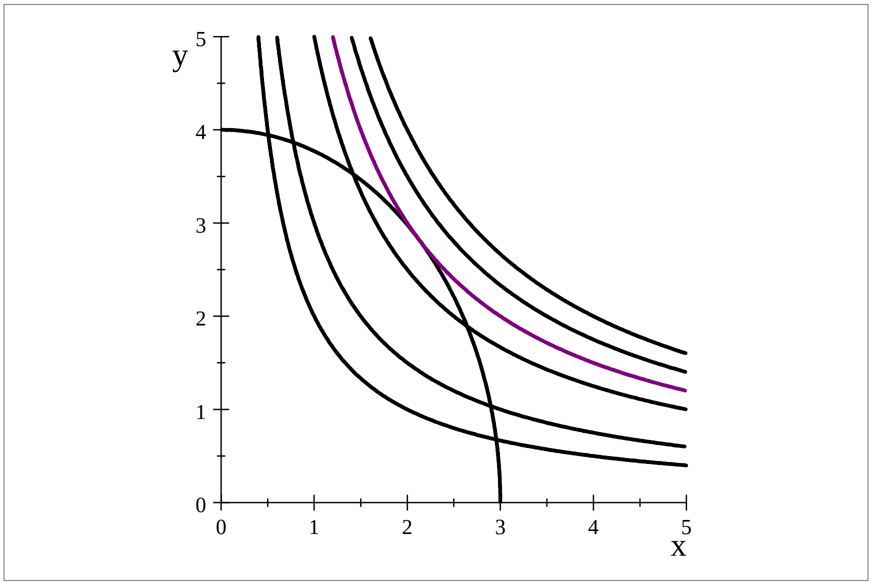


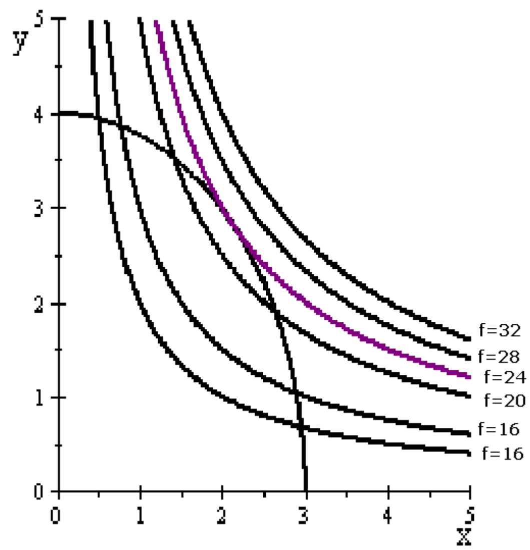
Let us consider the problems in terms of the level curves of the function $f(x,y)$ for various constants and its relationship with the graph of $\frac{x^2}{9} + \frac{y^2}{16} = 1$

The graphs of $f(x,y) = c$ will be various hyperbolae, as shown below



It will be good to restrict ourselves to the first quadarnt only, and note that





In the above example, we can note that the maximum value of $f(x,y) = 4xy$ subject to $\frac{x^2}{9} + \frac{y^2}{16} = 1$

occurs when the two graphs touch each other, or in other words they share a common tangent line and consequently the normal vectors to these two graphs are parallel to each other.

Remember that the gradient vector gives us a direction along the normal to a level curve, therefore we can use the gradient vector to find the values of x,y that will give such a maximum value.

If we took, $g(x,y) = \frac{x^2}{9} + \frac{y^2}{16}$

$$\nabla g = \left\langle \frac{2x}{9}, \frac{2y}{16} \right\rangle = \left\langle \frac{2x}{9}, \frac{y}{8} \right\rangle$$

we have

For $f(x,y) = 4xy$

$\nabla f = \langle 4y, 4x \rangle$

At the point of our interest, ∇f and ∇g are parallel to each other.

therefore we should be able to find a scalar λ such that

$\nabla f = \lambda \nabla g$

i.e.

$$\langle 4y, 4x \rangle = \lambda \left\langle \frac{2x}{9}, \frac{y}{8} \right\rangle$$

$$4y = \frac{2\lambda x}{9} \rightarrow y = \frac{\lambda x}{18}$$
$$4x = \frac{\lambda y}{8} \rightarrow x = \frac{\lambda y}{32}$$

substituting $y = \frac{\lambda x}{18}$ in $x = \frac{\lambda y}{32}$

we get

$$x = \frac{\lambda}{32}\left(\frac{\lambda x}{18}\right)$$

$$\rightarrow x = \frac{\lambda^2}{32 \times 18} x$$

Note that $x = 0$ is not a value that fits in our problem, therefore we can cancel x to obtain

$$1 = \frac{\lambda^2}{576}$$

OR

$$\lambda^2 = 576$$
$$\lambda = \pm\sqrt{576} = 24$$

Therefore we have

$$y = \frac{(24)x}{18} = \frac{4}{3}x$$

substitute this value in

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

to obtain

$$\frac{x^2}{9} + \frac{\left(\frac{4}{3}x\right)^2}{16} = 1$$

$$\frac{x^2}{9} + \frac{x^2}{9} = 1$$
$$2x^2 = 9$$
$$x^2 = \frac{9}{2}$$

$$x = \pm\frac{3}{\sqrt{2}}$$

$$y = \pm\frac{4}{3}\left(\frac{3}{\sqrt{2}}\right) = \pm 2\sqrt{2}$$

The corresponding points are $\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right), \left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right),$
$\left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right), \left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$

(x,y)	$f(x,y) = 4xy$
$\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$	24
$\left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$	–24
$\left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$	–24
$\left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$	24

The maximum value of $f(x,y)$ subject to the given constarint is 24.

You could easily have done the above problem using single variable calculus

....

Let us look at the extension of this method for higher dimensions

We would like to maximize or minimize a function $f(x,y,z)$ subject to the contraint $g(x,y,z) = c$

The method is based on the assumptions that both f, g have continuous partial derivatives, the exteme values exist, and that $\nabla g \neq 0$
on the surface $g(x, y, z) = c$

The method based on the Lagrange's Theorem (Theorem 13.19 in the text book)

The method works as follows:

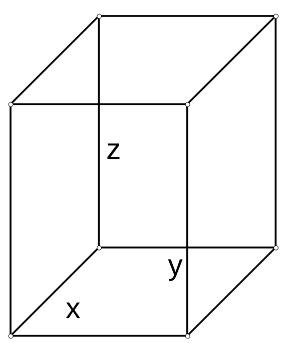
First find the values of x, y, z and λ such that

$\nabla f = \lambda \nabla g$
and $g(x, y, z) = c$

then evaluate the values of the function at the points obtained in the first step. Comapre the values that you obtain to find the maximum and the minimum.

Example:

We are given a cardboard with an area of 12 square feet and would like to construct a box with open top with maximum possible volume.



area of the base is xy
areas of the sides are xz,yz,xz,yz

NO TOP

Total area is 2xy+2yz+xy that equals
12 square feet

We have to maximize

$f(x, y, z) = xyz$

subject to

$xy + 2xz + 2yz = 12$

take $g(x,y,z) = xy + 2xz + 2yz$

then

we have to maximize
$f(x,y,z) = xyz$

subject to $g(x,y,z) = 12$
Find

$\nabla f = \langle yz, zx, xy \rangle$
$\nabla g = \langle y + 2z, x + 2z, 2x + 2y \rangle$

set

$\nabla f = \lambda \nabla g$
$\langle yz, zx, xy \rangle = \lambda \langle y + 2z, x + 2z, 2x + 2y \rangle$

$yz = \lambda(y + 2z)$ .... (1)
$zx = \lambda(x + 2z)$ .... (2)
$xy = \lambda(2x + 2y)$ .... (3)
and we also have

$xy + 2xz + 2yz = 12$ ....(4)

Now we have to skillfully work with these equations

First note that $\lambda \neq 0$ because his will make

$xy = yz = zx = 0$ and that will give the area of the available cardboard as 0

Observe that if we multiply the equation (1) by x
and the equation (2) by y, we shall get two identical terms to eliminate

$$yz = \lambda(y + 2z) \quad \rightarrow \quad xyz = \lambda xy + 2\lambda xz$$

$$zx = \lambda(x + 2z) \quad \rightarrow \quad yzx = \lambda xy + 2\lambda yz$$

Set

$$\lambda xy + 2\lambda xz = \lambda xy + 2\lambda yz$$

$$2\lambda xz = 2\lambda yz$$

since $\lambda \neq 0$

$xz = yz$, $z \neq 0$ beacuse that will give 0 as the volume

therefore

$$x = y$$

Now

$$zx = \lambda(x + 2z) \quad \rightarrow \quad yzx = \lambda xy + 2\lambda yz$$

$$xy = \lambda(2x + 2y) \quad \rightarrow \quad xyz = 2\lambda xz + 2\lambda yz$$

we have

$$\lambda xy + 2\lambda yz = 2\lambda xz + 2\lambda yz$$

OR

$$\lambda xy = 2\lambda xz$$

since

$\lambda \neq 0$ and $x \neq 0$

$$y = 2z$$

subsitute

$x = y$
and
$y = 2z$
in

$xy + 2xz + 2yz = 12$

to obtain

$(2z)(2z) + 2(2z)z + 2(2z)z = 12$

$12z^2 = 12 \rightarrow z^2 = 1$

$z = 1$ ft is applicable to our case
$x = y = 2$ feet

maximum volume is $2 \times 2 \times 1 = 4$ cubic feet

Example 2:

Let us work on #14 on the page 974

To maximize $f(x,y) = e^{-xy/4}$ subject to $x^2 + y^2 \leq 1$

In this case we are looking at two regions

PARTA: inside the circle $x^2 + y^2 = 1$

PARTB: the boundary of the circle $x^2 + y^2 = 1$

For PARTA

We shall use the method that we used to obtain the extrema on an open set by finding the critical points by setting the partial derivatives to 0

$f(x,y) = e^{-xy/4}$
$\rightarrow$
$\frac{\partial f}{\partial x} = -\frac{y}{4}e^{-xy/4}$
$\frac{\partial f}{\partial y} = -\frac{x}{4}e^{-xy/4}$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$
at $(x,y) = (0,0)$ because $e^{-xy/4} \neq 0$

PARTB:

$\nabla f = \left\langle -\frac{y}{4}e^{-xy/4}, -\frac{x}{4}e^{-xy/4} \right\rangle$
Take $g(x,y) = x^2 + y^2$
constraint is $g(x,y) = 1$
$\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$\rightarrow$
$\left\langle -\frac{y}{4}e^{-xy/4}, -\frac{x}{4}e^{-xy/4} \right\rangle = \lambda \langle 2x, 2y \rangle$
$-\frac{y}{4}e^{-xy/4} = 2\lambda x \rightarrow e^{-xy/4} = -8\frac{\lambda x}{y}$
$-\frac{x}{4}e^{-xy/4} = 2\lambda y \rightarrow e^{-xy/4} = -8\frac{\lambda y}{x}$

$\frac{\lambda x}{y} = \frac{\lambda y}{x}$
gives
$\frac{x}{y} = \frac{y}{x}$ because $\lambda \neq 0$

$x^2 = y^2$

$y = \pm x$

substitute in

$x^2 + y^2 = 1$
$x^2 + x^2 = 1$
$2x^2 = 1$
$x^2 = \frac{1}{2}$
$x = \pm\frac{1}{\sqrt{2}}$

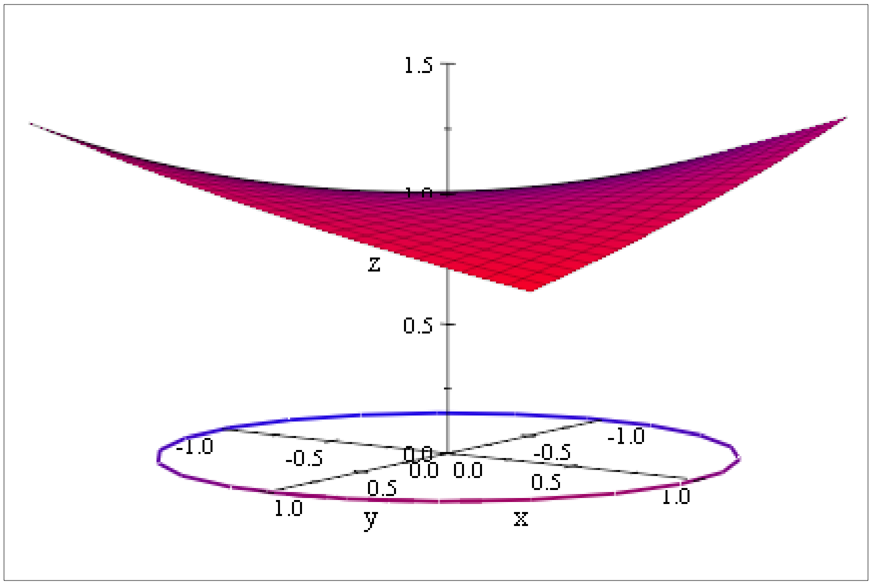
We shall compare the values at

$\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right),\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $(0,0)$

POINT	$f(x,y) = e^{-xy/4}$	
$\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$	$e^{-\left(-1/\sqrt{2}\right)\left(-1/\sqrt{2}\right)/4} = e^{-1/8}$	MIN
$\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	$e^{-\left(-1/\sqrt{2}\right)\left(1/\sqrt{2}\right)/4} = e^{1/8}$	MAX
$\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$	$e^{-\left(1/\sqrt{2}\right)\left(-1/\sqrt{2}\right)/4} = e^{1/8}$	MAX
$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	$e^{-\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right)/4} = e^{-1/8}$	MIN
$(0,0)$	$e^{-(0)(0)/4} = 1$	

You may apply the second derivative test to see that there is a saddle point at $(0,0,1)$

$e^{-xy/4}$



If we have two constarint functions g and h, we can introduce additional Lagrange Multiplier μ

solve

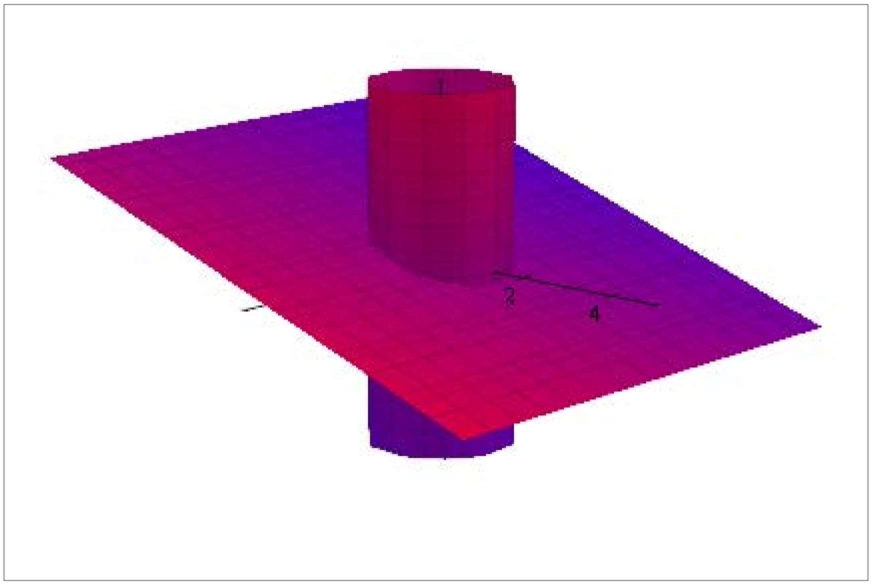
$\nabla f = \lambda \nabla g + \mu \nabla h$

and continue

Example 3:

To find the maximum and minimum values of $f(x,y,z) = -x + 2y + 2z$

constrained on the ellipse $x^2 + y^2 = 2, y + 2z = 1$



The problem is actually to

To find the maximum and minimum values of $f(x,y,z) = -x + 2y + 2z$

subject to two constrints
$x^2 + y^2 = 2, y + 2z = 1$

Take $g(x,y) = x^2 + y^2$
$h(x,y) = y + 2z$

$\nabla f = \langle -1, 2, 2 \rangle$
$\nabla g = \langle 2x, 2y, 0 \rangle$
$\nabla h = \langle 0, 1, 2 \rangle$

$\langle -1, 2, 2 \rangle = \lambda \langle 2x, 2y, 0 \rangle + \mu \langle 0, 1, 2 \rangle$

$2\lambda x = -1$
$2\lambda y + \mu = 2$
$2\mu = 2 \rightarrow \mu = 1$
$x^2 + y^2 = 2$

$y + 2z = 1$

$2\lambda y + \mu = 2 \to 2\lambda y + 1 = 2 \to 2\lambda y = 1 \to y = \frac{1}{2\lambda}$

$2\lambda x = -1 \to x = -\frac{1}{2\lambda}$

substitute in $x^2 + y^2 = 2$

to obtain

$\left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 2$

$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 2$

$\frac{1}{2\lambda^2} = 2$

$\lambda^2 = \frac{1}{4}$

$\lambda = \pm\frac{1}{2}$

for $\lambda = \frac{1}{2}$

$x = -\frac{1}{2\lambda} = -1$

$y = \frac{1}{2\lambda} = 1$

use these values in

$y + 2z = 1$ to obtain $z = 0$

the point corresponding to $\lambda = \frac{1}{2}$ is $(-1, 1, 0)$

for $\lambda = -\frac{1}{2}$

$x = -\frac{1}{2\lambda} = 1$

$y = \frac{1}{2\lambda} = -1$

use these values in

$y + 2z = 1$ to obtain $z = 1$

the point corresponding to $\lambda = \frac{1}{2}$ is $(1,-1,1)$

$f(-1,1,0) = -(-1) + 2(1) + 2(0) = 3$ max

$f(1,-1,1) = -(1) + 2(-1) + 2(1) = -1$ min

Suggested Practice Problems in the section 13.10

5,11,13,21,33,35