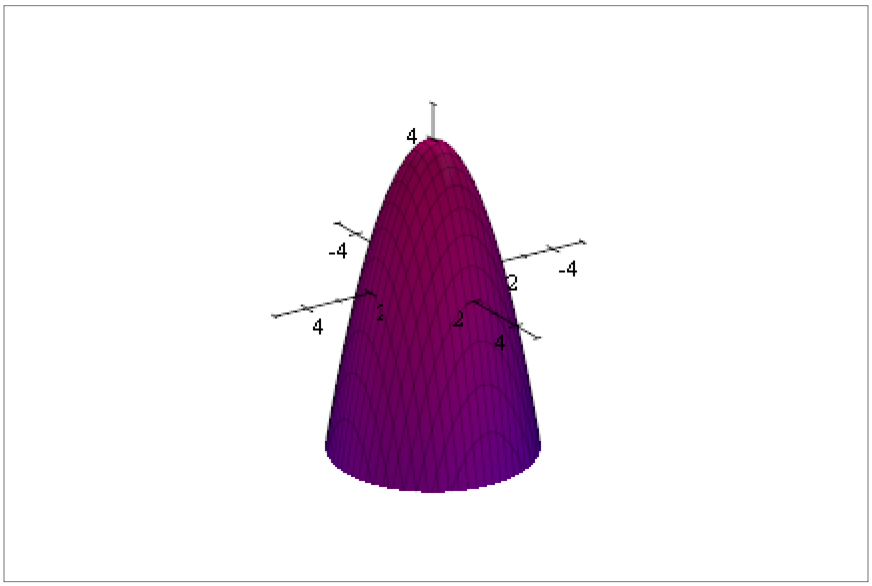
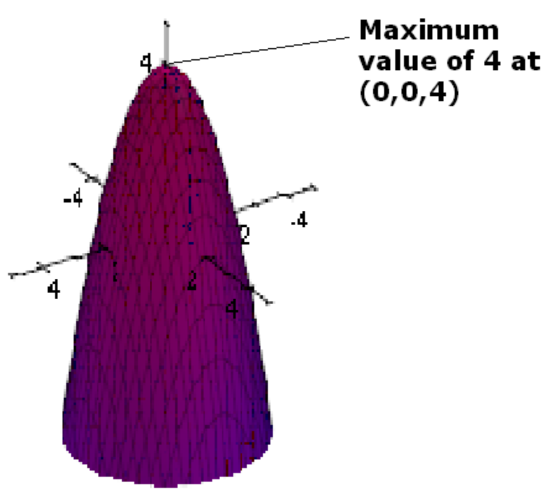
In this section, we are going to work on the maxima and minima of functions of two variables.

Example 1:

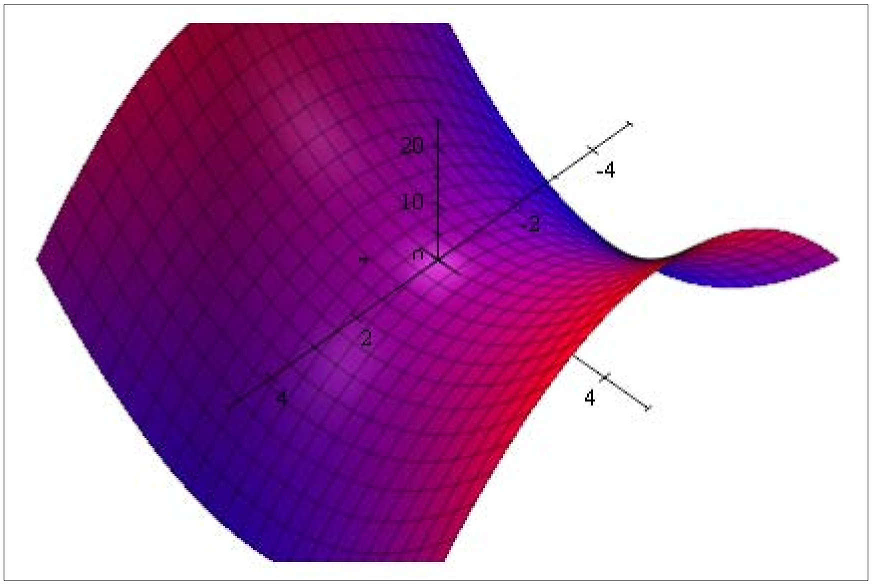
$f(x,y) = 4 - x^2 - y^2$

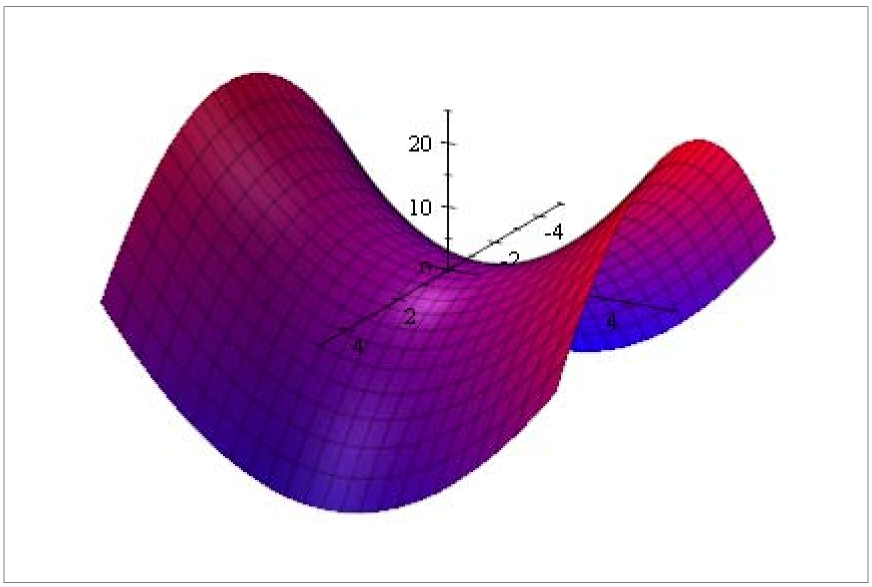




You can make out from the expression itself that the highest value of the function is 4 units and it does not have any minimum value.

Example 2:
$g(x,y) = y^2 - x^2$

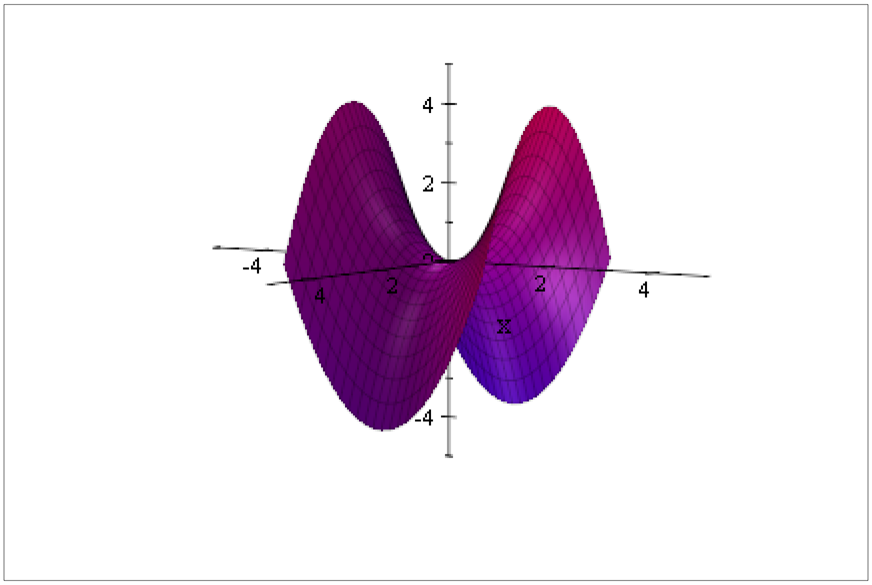


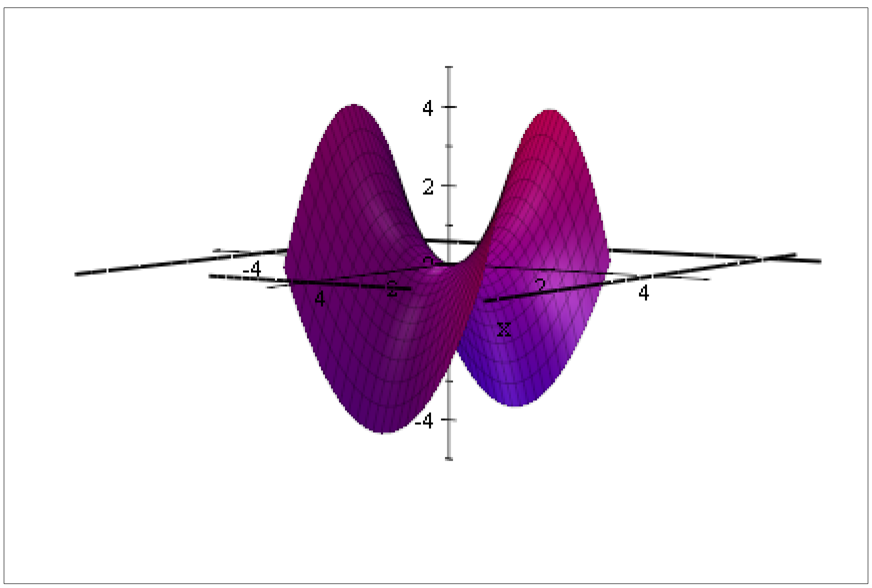


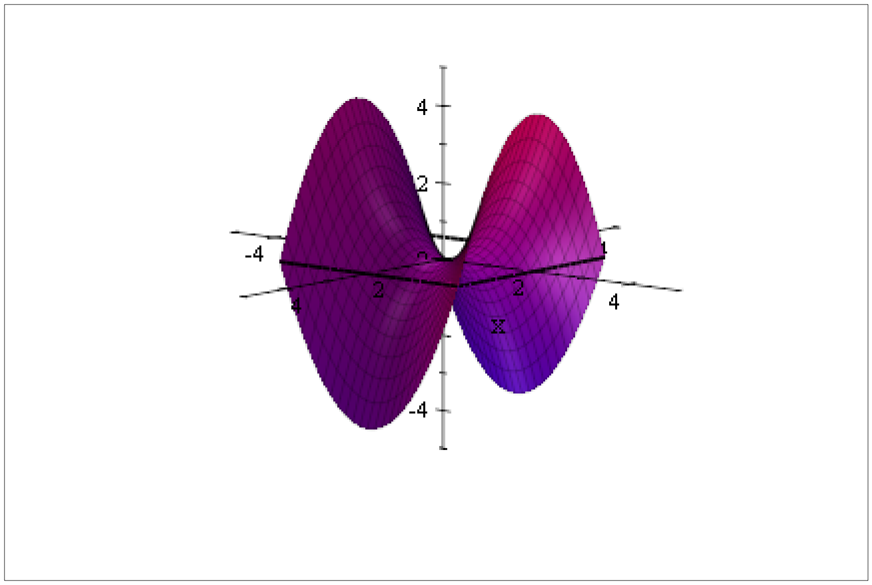
This function neither has a maximum nor a minimum.

Notice that the shape of the surface is like a saddle.

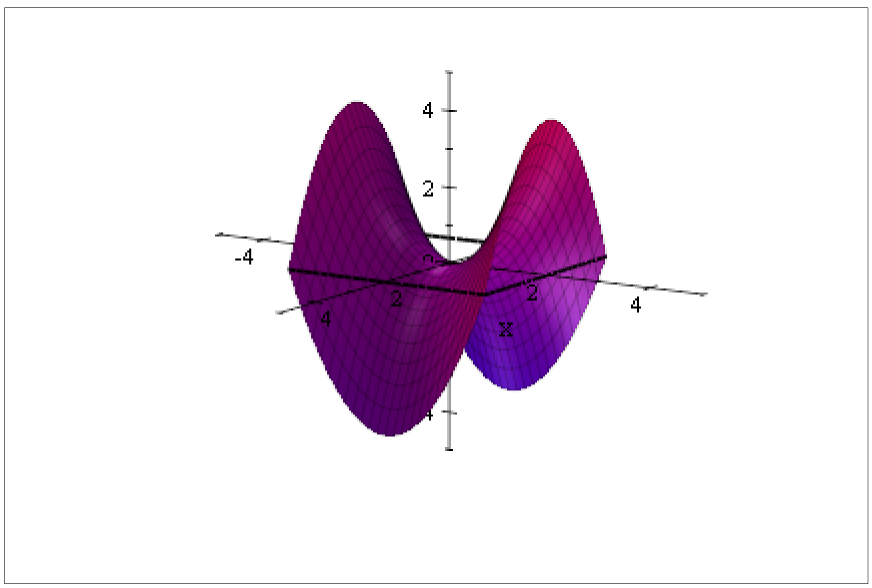
Example 3:
$g(x,y) = y^2 - x^2, \quad -2 \leq x \leq 2$ and $-2 \leq y \leq 2$


4
2
-4
4
2
2
4
x
-4

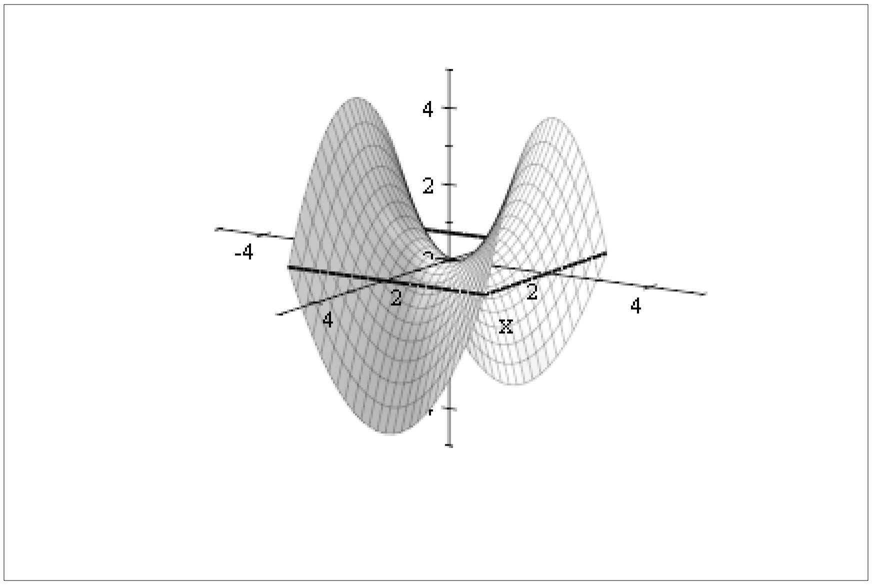

4
2
-4
4
2
2
4
x
-4

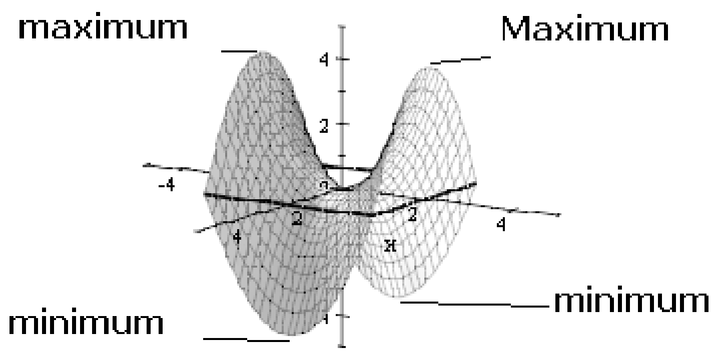


4
2
-4
2
2
4
x
-4



4
2
-4
2
2
4
x

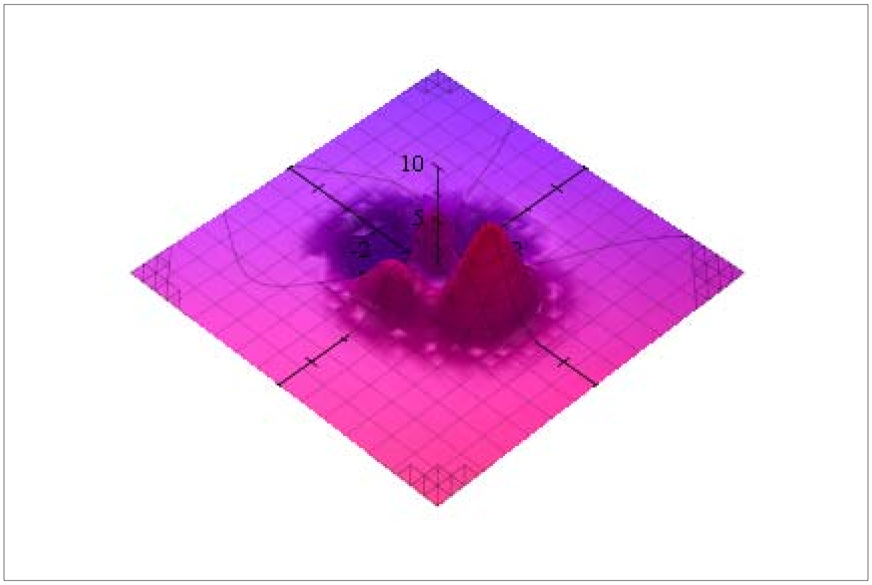


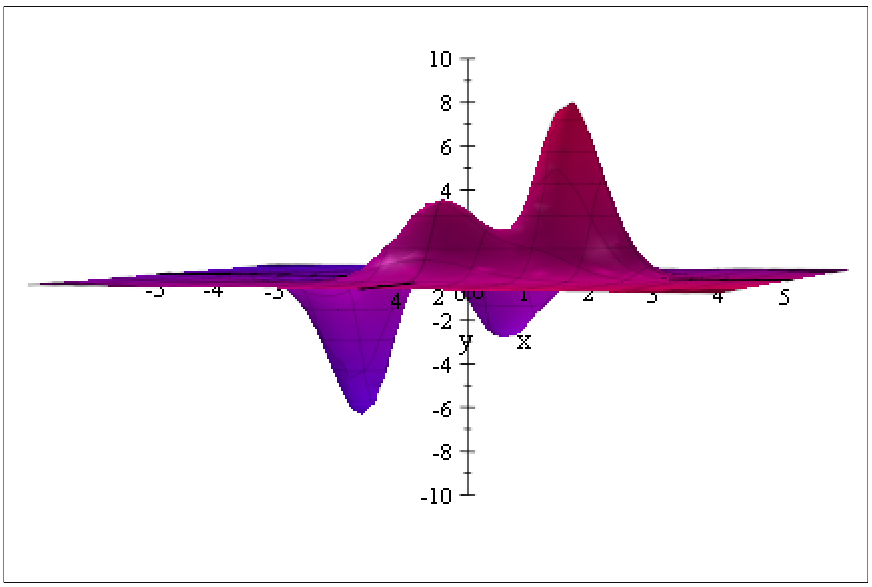


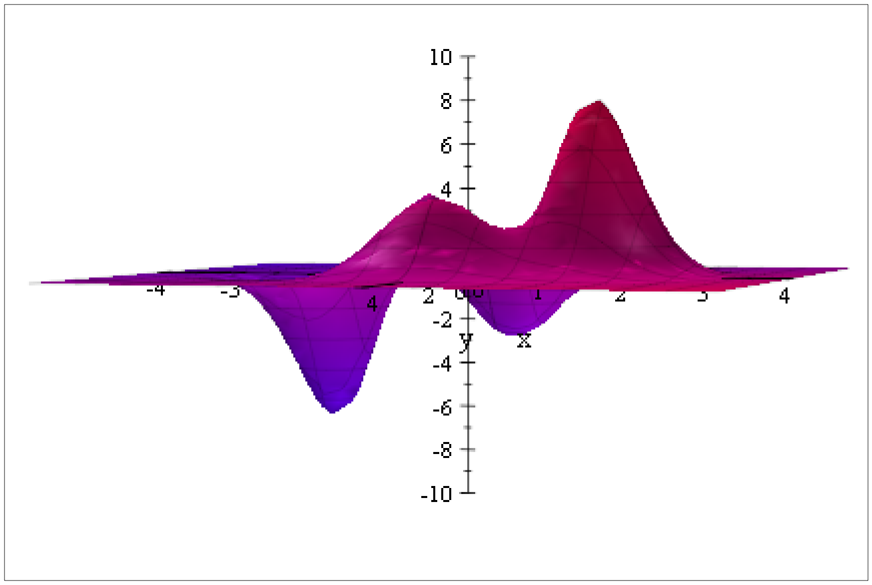
If the doamin of a continuous function is a closed and bounded region, the maximum and the minimum, both the values are available.

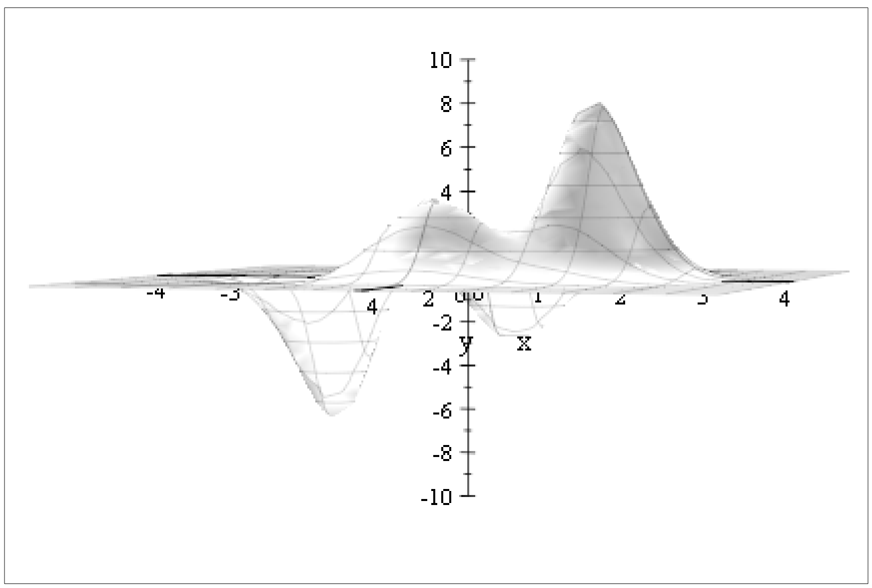
Example 4:

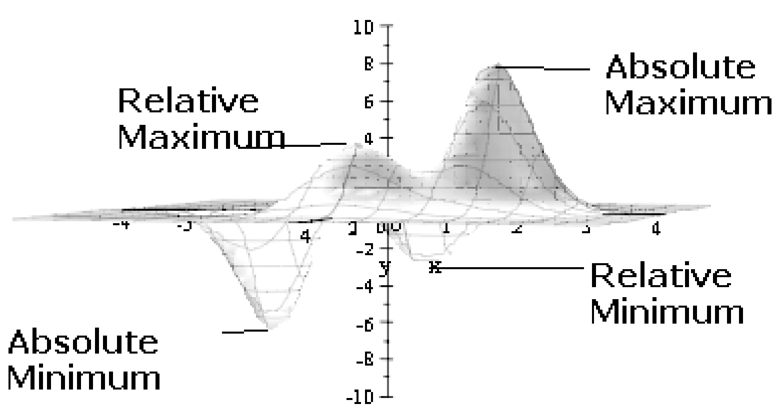
$$f(x,y) = 3(x-1)^2 e^{-x^2-(y+1)^2} - 10\left(\frac{1}{5}x - x^3 - y^5\right)e^{-x^2-y^2} - \frac{1}{3}e^{-(x+1)^2-y^2}$$


10
5


10
8
6
4
-2
-4
-6
-8
-10
-5
-4
-3
4
2
0
1
2
3
4
5
y
x


10
8
6
4
-2
-4
-6
-8
-10
-4
-3
4
2
1
2
3
4
y
x


10
8
6
4
-2
-4
-6
-8
-10
-4
-3
4
2
1
2
3
4
y
x


Absolute Maximum
Relative Maximum
Relative Minimum
Absolute Minimum
10
8
6
4
-2
-4
-6
-8
-10
y
x

If $f(x.y)$ is continuous on a closed and bounded region, then the absolute minimum or absolute maximum exist at a point such that

it is an interior point at which both the partial derivatives $\left(\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}\right)$ are zeros

OR

it is an interior point at which both one or both of the partial deriavtives $\left(\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}\right)$ are not available.

OR

a point on the boundary of the domain

The three types of pecial points described above are called the critical points of the function.

Example 1:

Let us find the absolute maximum and the absolute minimum for the function $f(x,y) = x^2 - 6x + y^2 - 8y + 7$ if this function is restricted in the region $\left\{(x,y) : x^2 + y^2 \leq 1\right\}$

$f(x,y) = x^2 - 6x + y^2 - 8y + 7$

Let us first find the critical points

$\frac{\partial f}{\partial x} = 2x - 6$

$$\frac{\partial f}{\partial y} = 2y - 8$$

$$\frac{\partial f}{\partial x} = 0 \to 2x - 6 = 0 \to x = 3$$
$$\frac{\partial f}{\partial y} = 0 \to 2y - 8 = 0 \to y = 4$$

But we can not use $(3,4)$ beacuse $(3,4)$ is not in the domain $\left\{(x,y) : x^2 + y^2 \le 1\right\}$ for this situation

The boundary is given by $x^2 + y^2 = 1$

Note that an easy way to restrict the function

$$f(x,y) = x^2 - 6x + y^2 - 8y + 7$$

along $x^2 + y^2 = 1$ to find the extrema is to use the parametric form

$x = \cos\theta \qquad y = \sin\theta$ where $0 \le \theta \le 1$

The problem now changes to finding the extrema for the function

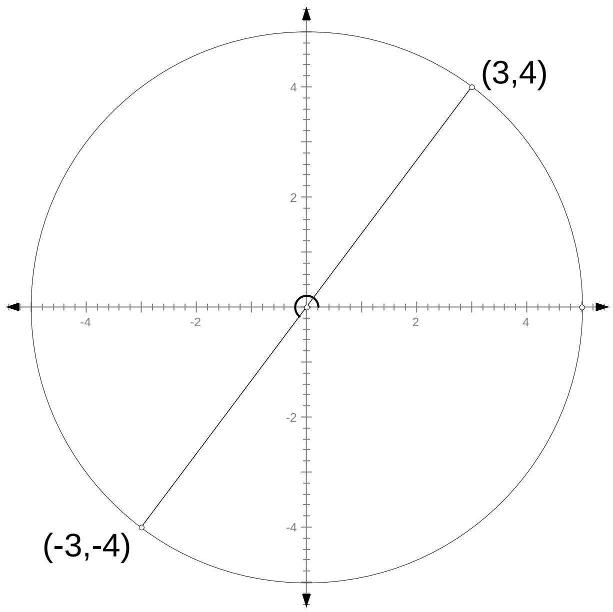
$$u(\theta) = (\cos\theta)^2 - 6\cos\theta + (\sin\theta)^2 - 8\sin\theta + 7$$
$$u(\theta) = (\cos\theta)^2 + (\sin\theta)^2 - 6\cos\theta - 8\sin\theta + 7$$
$$u(\theta) = 1 - 6\cos\theta - 8\sin\theta + 7$$
$$u(\theta) = 8 - 6\cos\theta - 8\sin\theta$$

$$\frac{du}{d\theta} = 6\sin\theta - 8\cos\theta$$

$$6\sin\theta - 8\cos\theta = 0 \to 6\sin\theta = 8\cos\theta \to \frac{\sin\theta}{\cos\theta} = \frac{8}{6} \to \tan\theta = \frac{4}{3}$$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$

can be in the first or the third quadratnt



$\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$ $\rightarrow$ $u(\theta) = 8 - 6\left(\frac{3}{5}\right) - 8\left(\frac{4}{5}\right) = -2$

$\cos\theta = -\frac{3}{5}$, $\sin\theta = -\frac{4}{5}$ $\rightarrow u(\theta) = 8 - 6\left(-\frac{3}{5}\right) - 8\left(-\frac{4}{5}\right) = 18$

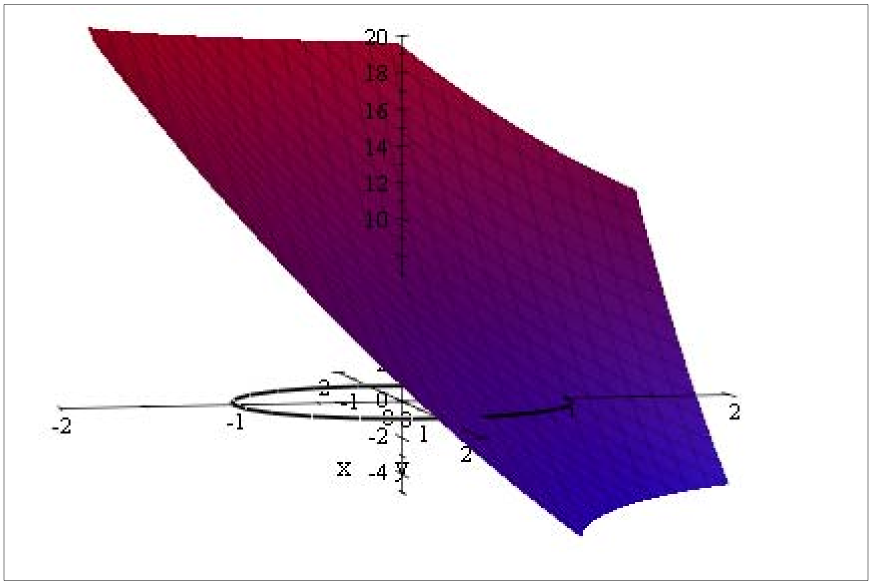
we also have to check the values at the end points of $[0, 2\pi]$

$u(0) = 8 - 6\cos 0 - 8\sin 0 = 2$

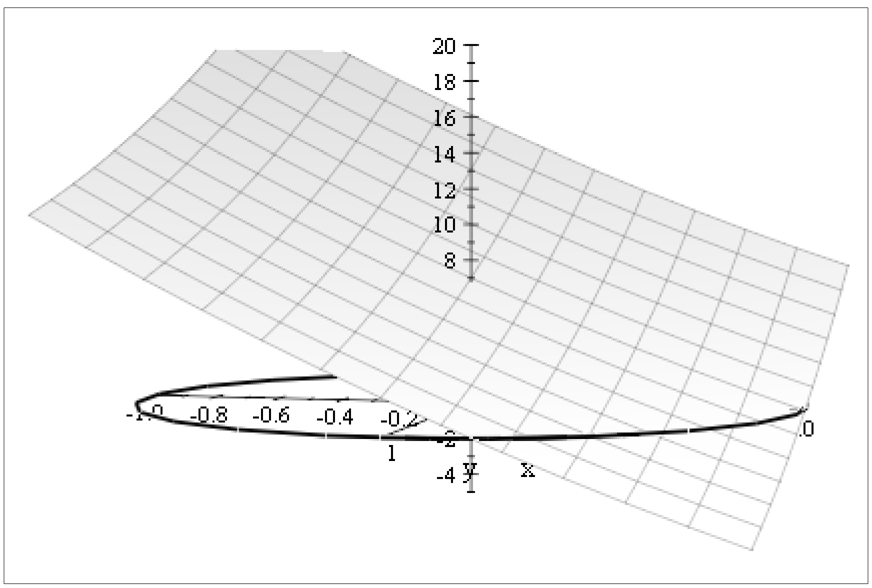
$u(2\pi) = 8 - 6\cos 2\pi - 8\sin 2\pi = 2$

Therefore the maximum value is 18, occurs at $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

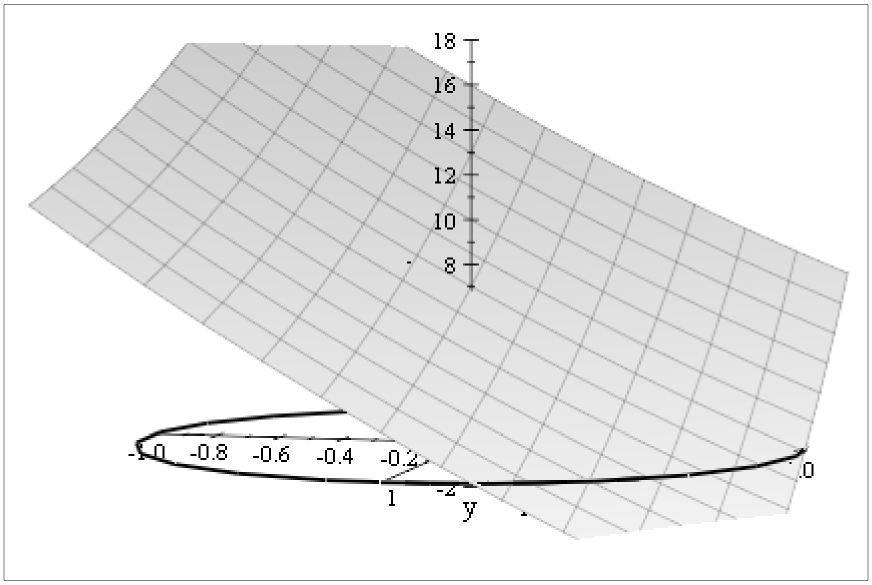
the minimum value is -2, occurs at $\left(\frac{3}{5}, \frac{4}{5}\right)$



20
18
16
14
12
10
-2
-1
2
x
-4

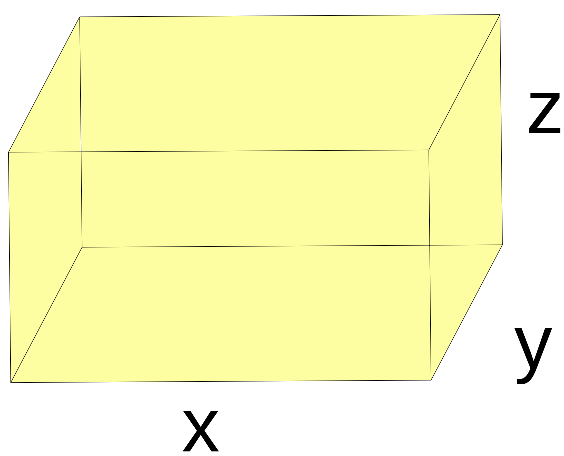


20
18
16
14
12
10
8
-0.8
-0.6
-0.4
1
-4
x



An application:

A certain airline requires that a cabin baggage(rectangular box shaped) should not have the sum of the length, width, and height should not be more than 120 cm. Find the largest volume that can be accomodated in a rectangular box that has the sum of its length



$x + y + z = 120$

To find the maximum value of the volume that is xyz
$x + y + z = 120 \rightarrow z = 120 - x - y$

$xyz = xy(120 - x - y)$

The function that we have to maximize is

$f(x,y) = xy(120 - x - y) = 120xy - x^2y - xy^2$

the domain of the function is a subset of

$\{(x,y) : 0 \le x \le 120, 0 \le y \le 120\}$

$$\frac{\partial f}{\partial x} = 120y - 2xy - y^2$$
$$\frac{\partial f}{\partial y} = 120x - x^2 - 2xy$$

Solve for

$$\frac{\partial f}{\partial x} = 120y - 2xy - y^2 = 0 \rightarrow 2xy = 120y - y^2 \quad \text{....... (1)}$$
$$\frac{\partial f}{\partial y} = 120x - x^2 - 2xy = 0 \rightarrow 2xy = 120x - x^2 \text{....... (2)}$$

$120y - y^2 = 120x - x^2$
$\rightarrow (120 - y)y = (120 - x)x$
$\rightarrow y = x$ OR $120 - y = x$

Note that $120 - y = x \rightarrow 120 = x + y$, which makes $f(x,y) = xy(120 - x - y) = 0$ and can not be a maximum value

Substitute $y = x$ in any of the equations (1) or (2)

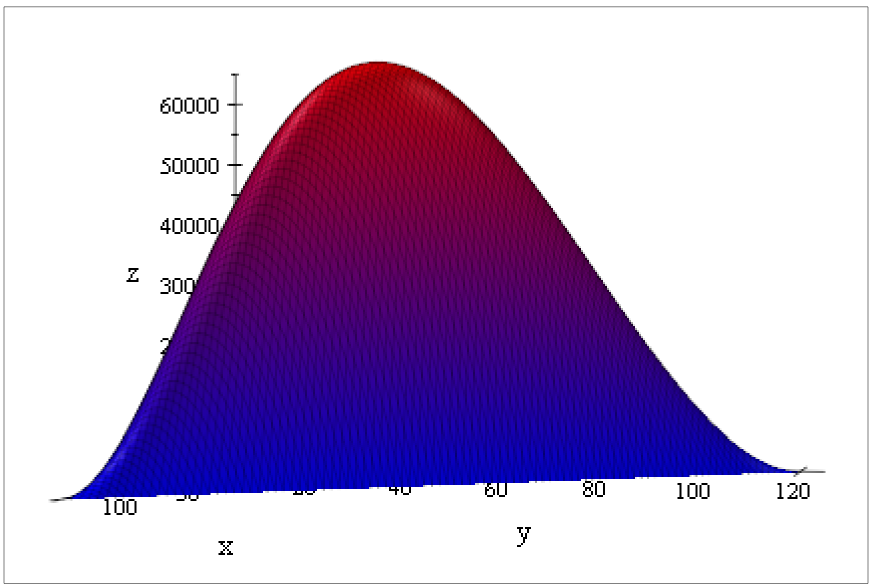
Let us substitute $y = x$ in (1)

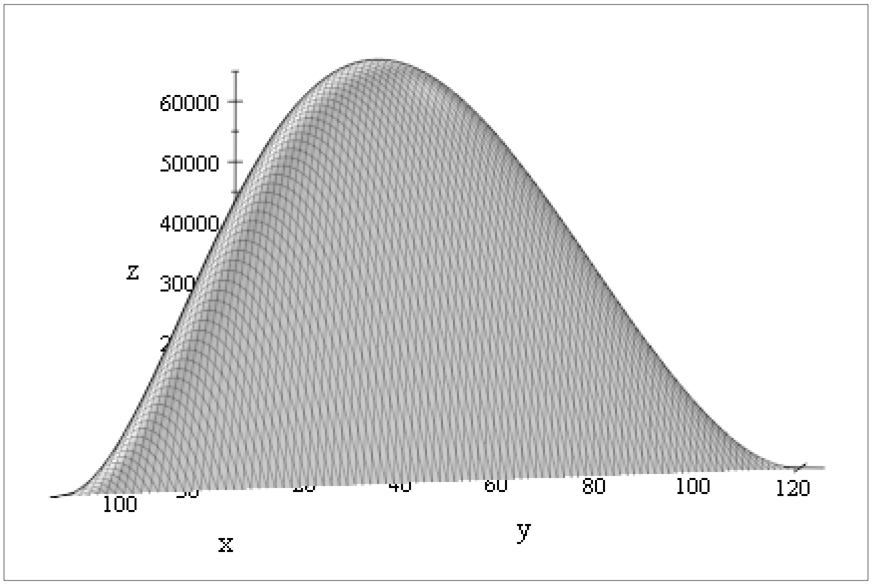
$2xx = 120x - x^2$
$\rightarrow 2x^2 = 120x - x^2$
$\rightarrow 3x^2 = 120x$
$\rightarrow 3x^2 - 120x = 0$
$\rightarrow 3x(x - 40) = 0$
$\rightarrow x = 0$ OR $x = 40$ cm

$x = 0$ will not give us maximum volume because this makes $f(x,y) = xy(120 - x - y)$ equal to zero

therefore $x = 40cm$, $y = 40cm$, $z = 40cm$

gives us a volume of
$f(40,40) = (40)(40)(120 - 40 - 40) = 64000 \ \ cm^3$





The second dervative test for a function $f(x,y)$ that has continuous partial derivatives in an open disk containing (a,b)

If $\nabla f\big|_{(a,b)} = 0$

and let

$$D = \left(\frac{\partial^2 f}{\partial x^2}\bigg|_{(a,b)} \right)\left(\frac{\partial^2 f}{\partial y^2}\bigg|_{(a,b)} \right) - \left(\left(\frac{\partial^2 f}{\partial x \partial y} \right)\bigg|_{(a,b)} \right)^2$$

Then

i) If $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2}\bigg|_{(a,b)} < 0$, then $f(a,b)$ is a relative maximum value

ii) If $D > 0$ and $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(a,b)} > 0$, then $f(a,b)$ is a relative minimum value

iii) If $D < 0$, then $f(a,b)$ is neither a maximum nor a minimum value but (a,b) is a saddle point.

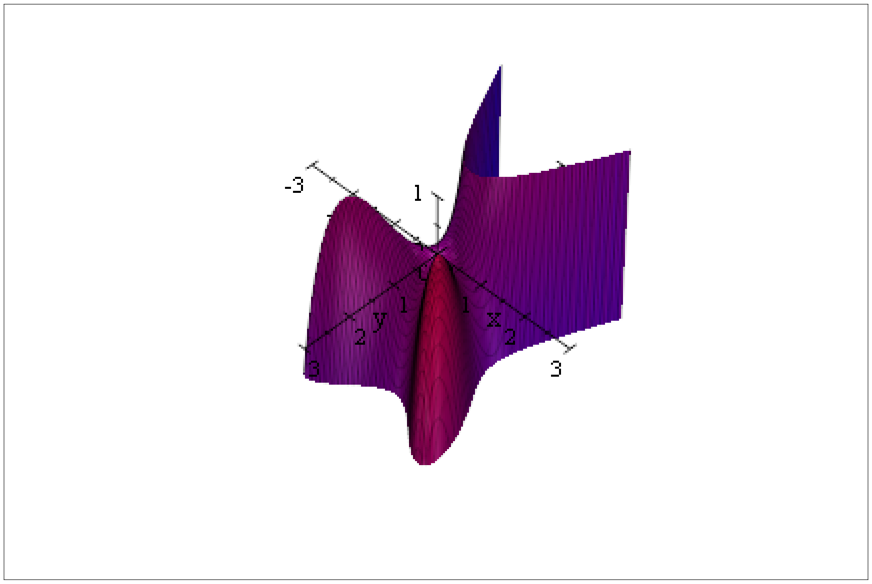
iv) If $D = 0$ the test is inconclusive

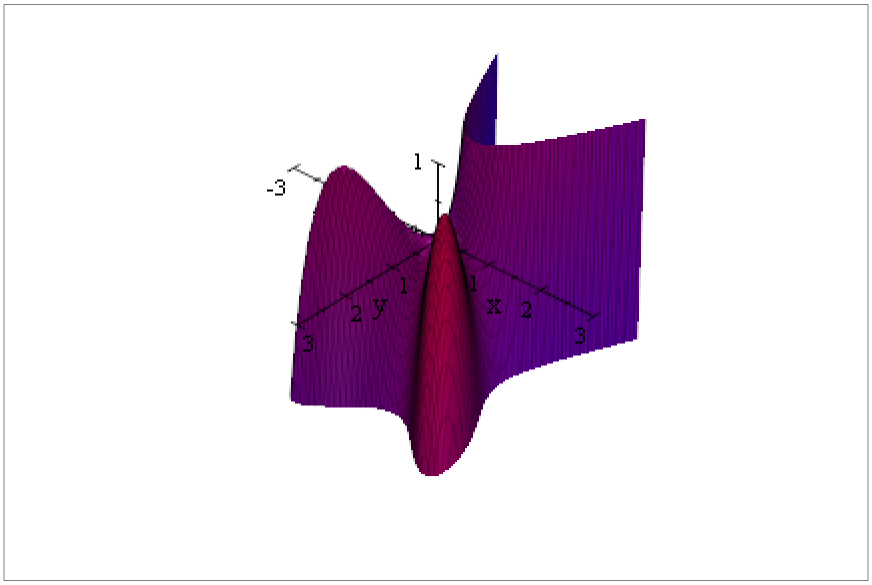
Example 3:

Let us use the second derivative test to find and classify the critical points for the

function

$f(x,y) = 6xy^2 - 2x^3 - 3y^4$





First, note that f is a differentiable function

$$\frac{\partial f}{\partial x} = 6y^2 - 6x^2$$
$$\frac{\partial f}{\partial y} = 12xy - 12y^3$$
$$\frac{\partial f}{\partial x} = 0 \to 6y^2 - 6x^2 = 0 \to y^2 = x^2 \to y = \pm x$$
$$\frac{\partial f}{\partial y} = 12xy - 12y^3 \to 12xy - 12y^3 = 0 \to 12y\left(x - y^2\right) = 0 \to y = 0 \text{ OR } x = y^2$$

$y = 0$ with $y = \pm x$ gives $(0,0)$ as a critical point

$x = y^2$ with $y = x$ gives, $y = y^2 \to y = 0$ OR $y = 1$ giving $(1,1)$ as a critical point

$x = y^2$ with $y = -x$ gives, $y = -y^2 \to y = 0$ OR $y = -1$ giving $(1,-1)$ as a critical point

Now, we would like to apply the second derivative test to these critical points.

$$\frac{\partial f}{\partial x} = 6y^2 - 6x^2 \rightarrow \frac{\partial^2 f}{\partial x^2} = -12x$$
$$\frac{\partial f}{\partial y} = 12xy - 12y^3 \rightarrow \frac{\partial^2 f}{\partial y^2} = 12x - 36y^2$$
$$\frac{\partial f}{\partial y} = 12xy - 12y^3 \rightarrow \frac{\partial^2 f}{\partial x \partial y} = 12y$$

For $(0, 0)$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = 0$$
$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 0$$
$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = 0$$

$D_{(0,0)} = (0)(0) - (0)^2 = 0$ The test Fails, we shall check this out later

For $(1, 1)$

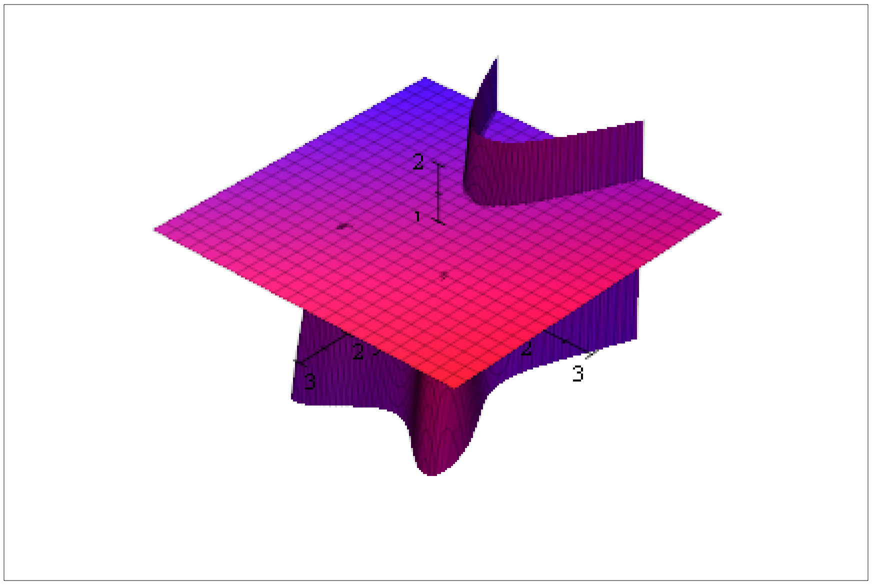
$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,1)} = -12(1) = -12$$
$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(1,1)} = 12(1) - 36(1)^2 = -24$$
$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(1,1)} = 12(1) = 12$$

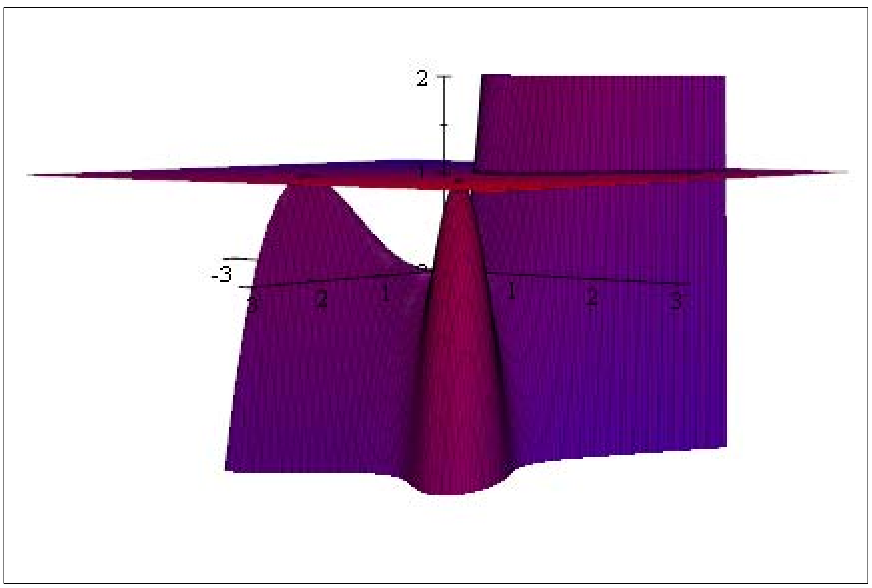
$$D_{(1,1)} = (-12)(-24) - 12^2 = 144 > 0$$
$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,1)} = -12$$
$$f(1, 1) = 6(1)(1)^2 - 2(1)^3 - 3(1)^4 = 1$$

$(1,1,1)$ Local Max

Similar calculations will give you that

$(1,-1,1)$ is also Local Max

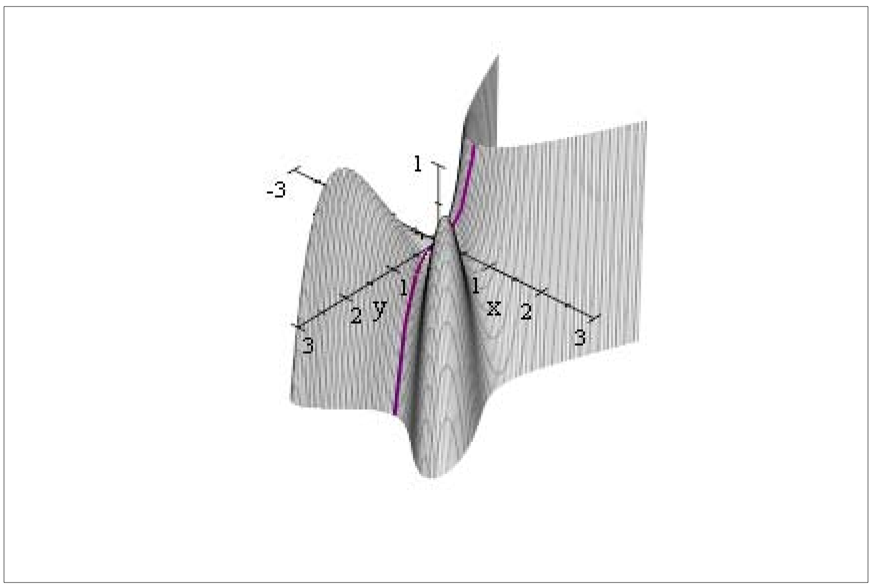


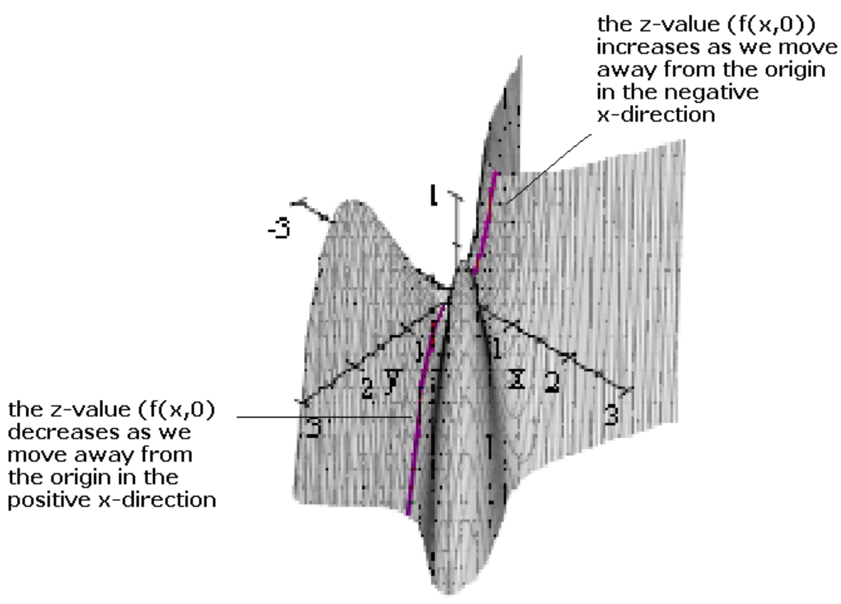


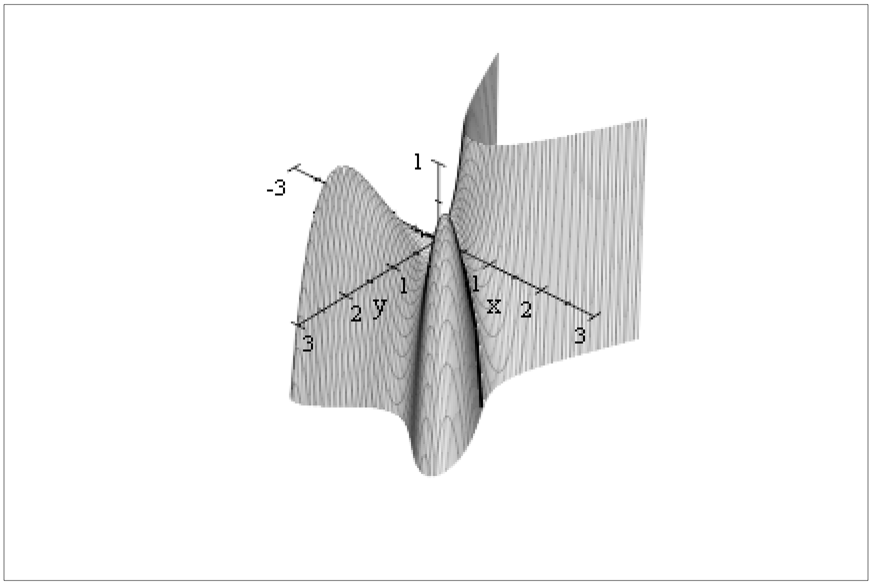
Now let us do some further study of the situation at $(0, 0, 0)$
Have

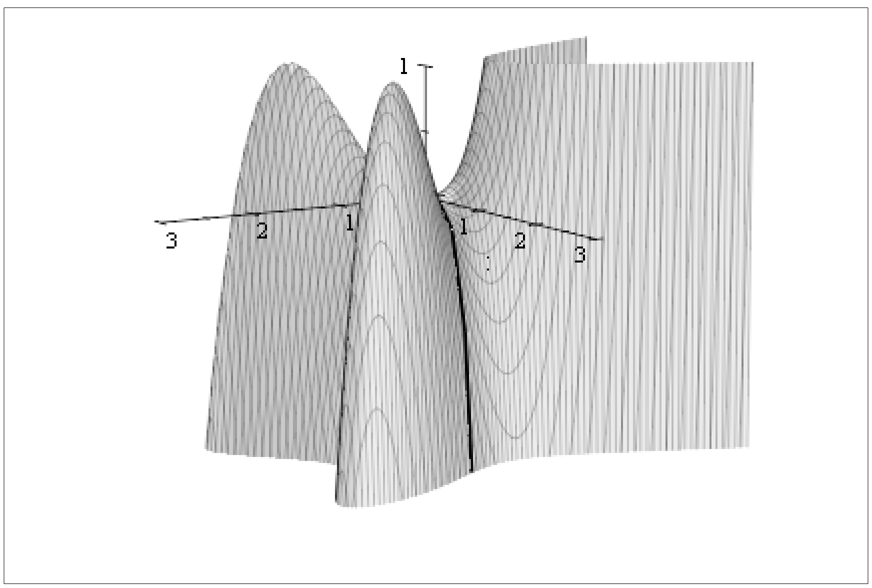
$$f(x,y) = 6xy^2 - 2x^3 - 3y^4$$

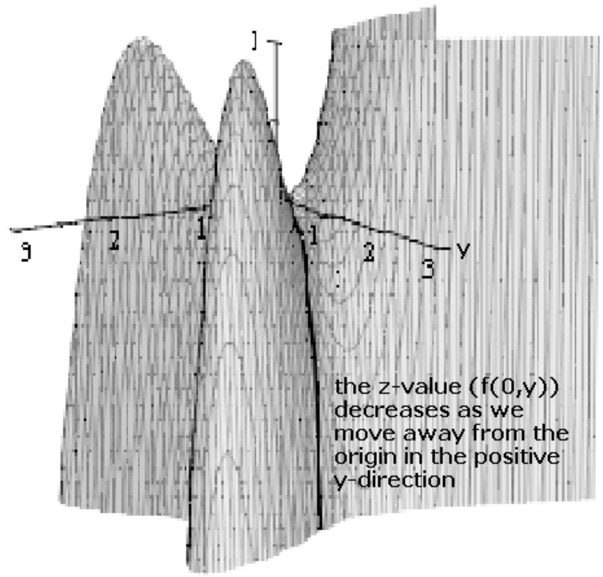
Consider
$f(x,0) = 6x(0)^2 - 2x^3 - 3(0)^4$
$f(x,0) = -2x^3$

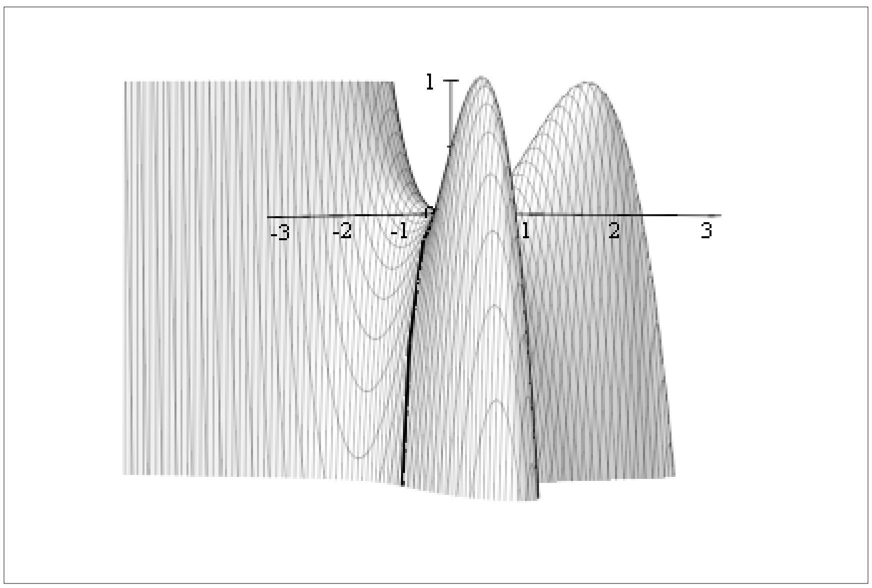


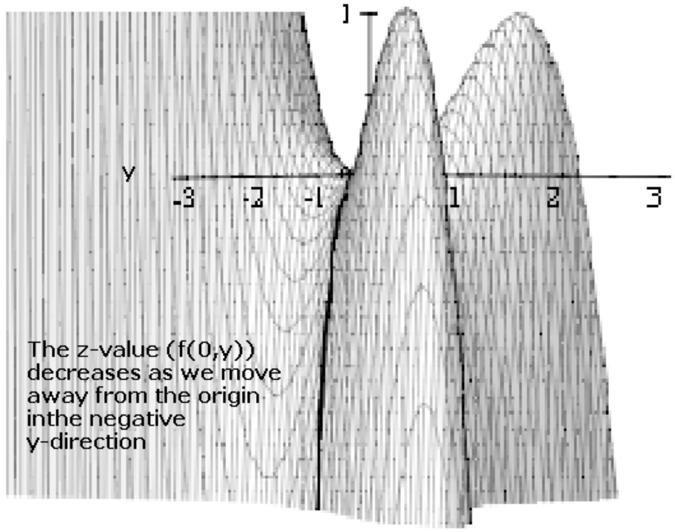



1
-3
1
2
y
3
1
x
2
3

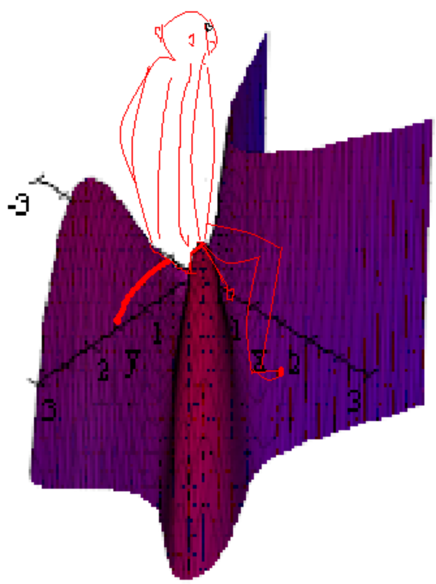

1
3
2
1
1
2
3


1
3
2
1
1
2
3
y
the z-value (f(0,y))
decreases as we
move away from the
origin in the positive
y-direction


1
-3
-2
-1
1
2
3



Therefore at $(0, 0, 0)$, neither we have a maximum nor a minimum.
Such a surface is often called a Monkey Saddle, because it provides room for a monkey to place its tail.



A very nice application is described in the Theorem 13.18 on the page 961 of the text book.

OPTIONAL:

You may watch the video

"A tale of the peacock’s Tail" that I have posted on the website

http://www.mathgurukul.org/Videotable.htm

Suggested Practice Problems:

Section 13.8:

15, 25, 27, 31,33,35, 43, 53, 55, 61, 63

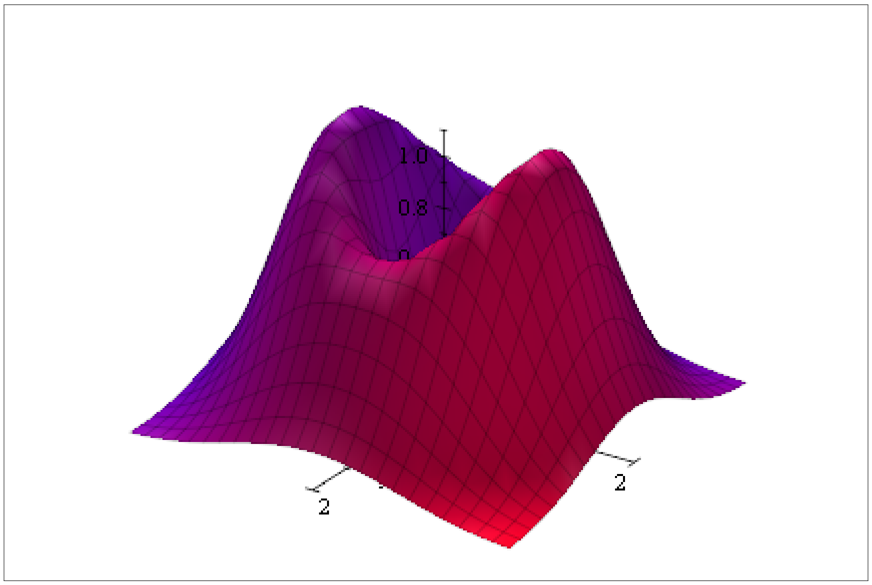
Section 13.9:

5, 7, 11, 15, 19, 21

Example 4:

To find the relative extrema of

$f(x,y) = \left(2x^2 + 3y^2\right)e^{-\left(x^2+y^2\right)}$ Note that the function is differentiable for all values of x and y



$$f(x,y) = \left(2x^2 + 3y^2\right)e^{-\left(x^2+y^2\right)}$$

$$\frac{\partial f}{\partial x} = 4xe^{-\left(x^2+y^2\right)} + \left(2x^2 + 3y^2\right)(-2x)e^{-\left(x^2+y^2\right)}$$

or

$$\frac{\partial f}{\partial x} = \left(4x - \left(2x^2 + 3y^2\right)2x\right)e^{-\left(x^2+y^2\right)}$$

or

$$\frac{\partial f}{\partial x} = \left(4x - 4x^3 - 6y^2x\right)e^{-\left(x^2+y^2\right)}$$

$$\frac{\partial f}{\partial y} = 6ye^{-\left(x^2+y^2\right)} - \left(2x^2 + 3y^2\right)(2y)e^{-\left(x^2+y^2\right)}$$

OR

$$\frac{\partial f}{\partial y} = \left(6y - 2y\left(2x^2 + 3y^2\right)\right)e^{-\left(x^2+y^2\right)}$$

OR

$$\frac{\partial f}{\partial y} = \left(6y - 4yx^2 - 6y^3\right)e^{-\left(x^2+y^2\right)}$$

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

implies

$$\left(4x - \left(4x^2 + 6y^2\right)x\right)e^{-\left(x^2+y^2\right)} = 0 \rightarrow 4x - 4x^3 - 6y^2x = 0 \rightarrow 2x\left(2 - 2x^2 - 3y^2\right) = 0$$

$$\left(6y - 4yx^2 - 6y^3\right)e^{-\left(x^2+y^2\right)} = 0 \rightarrow 6y - 4yx^2 - 6y^3 = 0 \rightarrow 2y\left(3 - 2x^2 - 3y^2\right) = 0$$

implies

$x\left(2 - 2x + 6y^2\right) = 0 \rightarrow x = 0$ OR $2 - 2x^2 - 3y^2 = 0$

$y\left(3y - 2x^2 - 3y^2\right) = 0 \rightarrow y = 0$ OR $3 - 2x^2 - 3y^2 = 0$

$x = 0, y = 0$

OR

$x = 0$, $3 - 2x^2 - 3y^2 = 0 \rightarrow 3 - 2(0)^2 - 3y^2 = 0 \rightarrow 3 - 3y^2 = 0 \rightarrow y = \pm 1$

$(0,0), (0,-1), (0,1)$

OR

$y = 0,\ 2 - 2x^2 - 3y^2 = 0 \rightarrow 2 - 2x^2 - 3(0)^2 = 0 \rightarrow x = \pm 1$

$(0,0), (-1,0), (1,0)$

the critical values of (x,y) for extema are $(0,0), (-1,0), (1,0), (0,-1), (0,1)$

Let us look at the second derivative test to determine, the points at which we have relative MIN or Relative MAX

Have $\frac{\partial f}{\partial x} = \left(4x - 4x^3 - 6y^2x\right)e^{-\left(x^2+y^2\right)}$, $\frac{\partial f}{\partial y} = \left(6y - 4yx^2 - 6y^3\right)e^{-\left(x^2+y^2\right)}$

$$\frac{\partial^2 f}{\partial x^2} = \left(4 - 12x^2 - 6y^2\right)e^{-\left(x^2+y^2\right)} + \left(4x - 4x^3 - 6y^2x\right)(-2x)e^{-\left(x^2+y^2\right)}$$

OR

$$\frac{\partial^2 f}{\partial x^2} = \left(4 - 12x^2 - 6y^2\right)e^{-\left(x^2+y^2\right)} + \left(4x - 4x^3 - 6y^2x\right)(-2x)e^{-\left(x^2+y^2\right)}$$

OR

$$\frac{\partial^2 f}{\partial x^2} = \left[4 - 12x^2 - 6y^2 - 8x^2 + 8x^4 + 12y^2x^2\right]e^{-\left(x^2+y^2\right)}$$

OR

$$\frac{\partial^2 f}{\partial x^2} = \left[4 - 20x^2 - 6y^2 + 8x^4 + 12y^2x^2\right]e^{-\left(x^2+y^2\right)}$$

$$\frac{\partial f}{\partial y} = \left(6y - 4yx^2 - 6y^3\right)e^{-\left(x^2+y^2\right)}$$

$$\frac{\partial^2 f}{\partial y^2} = \left(6 - 4x^2 - 18y^2\right)e^{-\left(x^2+y^2\right)} - \left(\left(6y - 4yx^2 - 6y^3\right)2y\right)e^{-\left(x^2+y^2\right)}$$

OR

$$\frac{\partial^2 f}{\partial y^2} = \left[6 - 4x^2 - 18y^2 - 12y^2 + 8y^2x^2 + 12y^4\right]e^{-\left(x^2+y^2\right)}$$

OR

$$\frac{\partial^2 f}{\partial y^2} = \left[6 - 4x^2 - 30y^2 + 8y^2x^2 + 12y^4\right]e^{-\left(x^2+y^2\right)}$$

$$\frac{\partial f}{\partial y} = \left(6y - 4yx^2 - 6y^3\right)e^{-\left(x^2+y^2\right)}$$

$\rightarrow$

$$\frac{\partial^2 f}{\partial x \partial y} = (-8xy)e^{-\left(x^2+y^2\right)} + \left(6y - 4yx^2 - 6y^3\right)(-2x)e^{-\left(x^2+y^2\right)}$$

$\rightarrow$

$$\frac{\partial^2 f}{\partial x \partial y} = \left[-8xy - 12xy + 8x^3y + 12xy^3\right]e^{-\left(x^2+y^2\right)}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(2x^2 + 3y^2\right)e^{-\left(x^2+y^2\right)}\right) = 4xye^{-x^2-y^2}\left(2x^2 + 3y^2 - 5\right)$$

Let us collect the steps together

$$\frac{\partial^2 f}{\partial x^2} = \left[4 - 20x^2 - 6y^2 + 8x^4 + 12y^2x^2\right]e^{-\left(x^2+y^2\right)}$$

$$\frac{\partial^2 f}{\partial y^2} = \left[6 - 4x^2 - 30y^2 + 8y^2x^2 + 12y^4\right]e^{-\left(x^2+y^2\right)}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \left[-20xy + 8x^3y + 12xy^3\right]e^{-\left(x^2+y^2\right)}$$

Remember that

$$D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

and that for this question the critical valus of (x, y) for extrema $(0,0), (-1,0), (1,0), (0,-1), (0,1)$

At $(0,0)$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = \left[4 - 20(0)^2 - 6(0)^2 + 8(0)^4 + 12(0)^2(0)^2\right]e^{-\left(0^2+0^2\right)} = 4$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = \left[6 - 4(0)^2 - 30(0)^2 + 8(0)^2(0)^2 + 12(0)^4\right]e^{-\left(0^2+0^2\right)} = 6$$

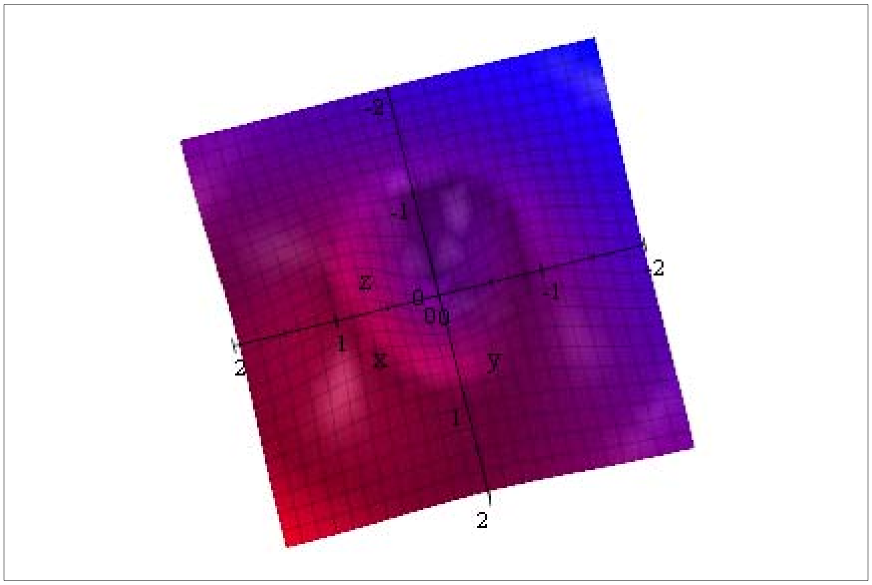
$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = \left[-20(0)(0) + 8(0)^3(0) + 12(0)(0)^3\right]e^{-\left(0^2+0^2\right)} = 0$$

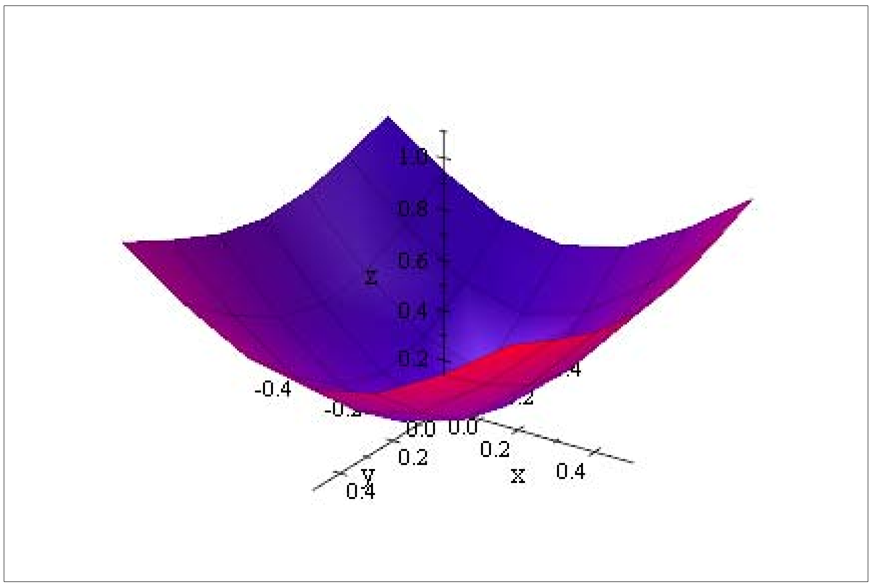
$D_{(0,0)} = 4 \times 6 - (0)^2 = 24 > 0$ and $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = 4 > 0$

Therefore there is a relative minimum at the point corresponding to $(0,0)$
or a relative minimum at $(0,0,)$

$$f(0,0) = \left(2(0)^2 + 3(0)^2\right)e^{-\left(0^2+0^2\right)} = 0$$

$\boxed{\text{there is a relative minimum at } (0,0,0)}$





At $(1,0)$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,0)} = \left[4-20(1)^2-6(0)^2+8(1)^4+12(0)^2(1)^2\right]e^{-\left(1^2+0^2\right)} = -8e^{-1}$$

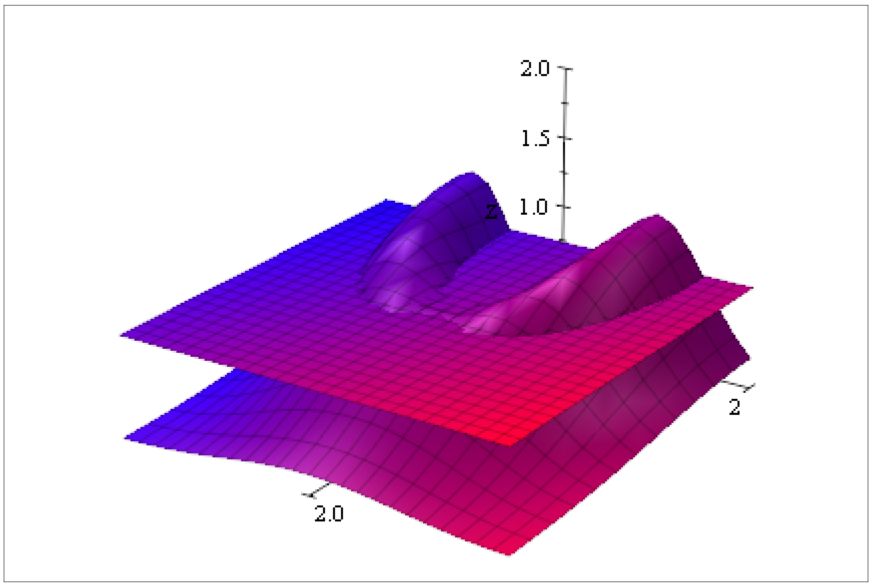
$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(1,0)} = \left[6-4(1)^2-30(0)^2+8(0)^2(1)^2+12(0)^4\right]e^{-\left(1^2+0^2\right)} = 2e^{-1}$$

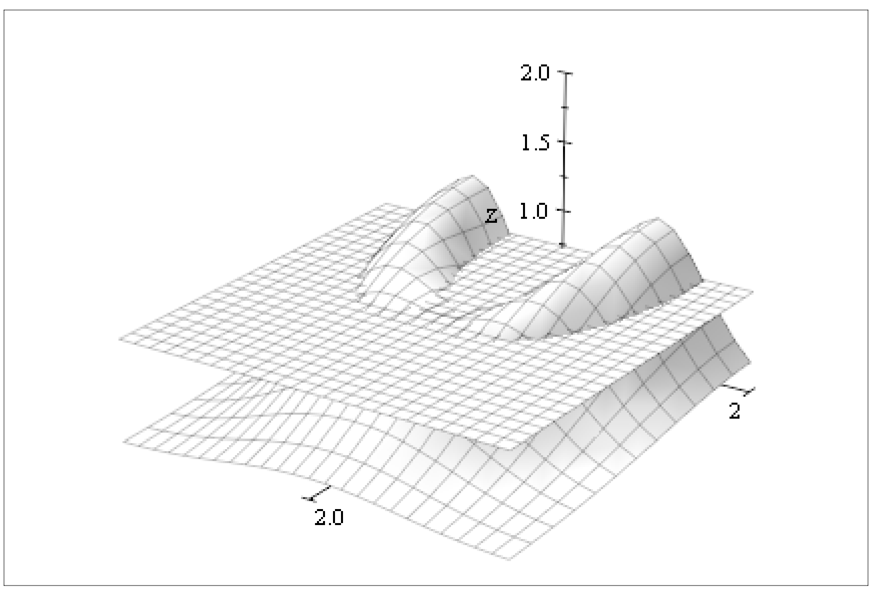
$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(1,0)} = \left[-20(1)(0) + 8(1)^3(0) + 12(1)(0)^3\right]e^{-\left(1^2+0^2\right)} = 0$$

$$D_{(1,0)} = \left(-8e^{-1}\right)\left(2e^{-1}\right) - 0^2 = -16e^{-2} < 0$$

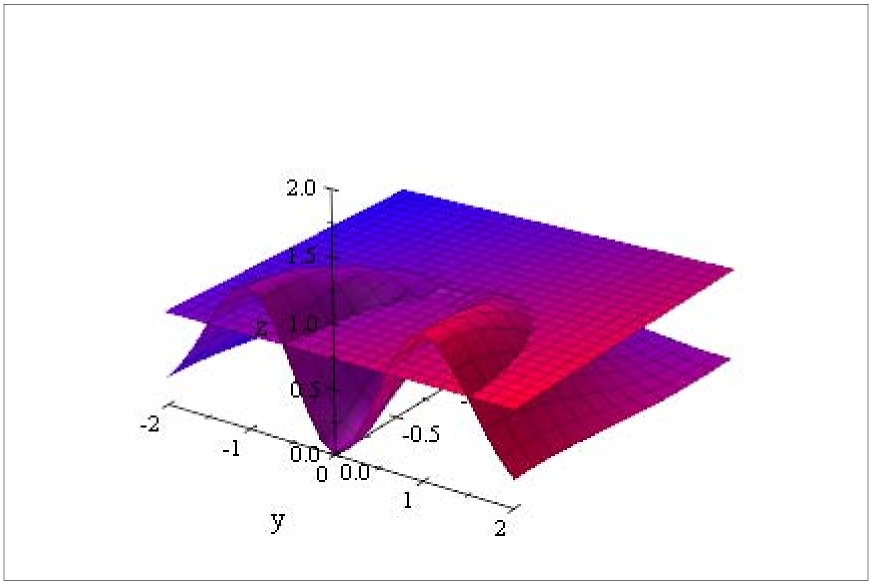
$$f(1,0) = \left(2(1)^2 + 3(0)^2\right)e^{-\left(1^2+0^2\right)} = 2e^{-1}$$

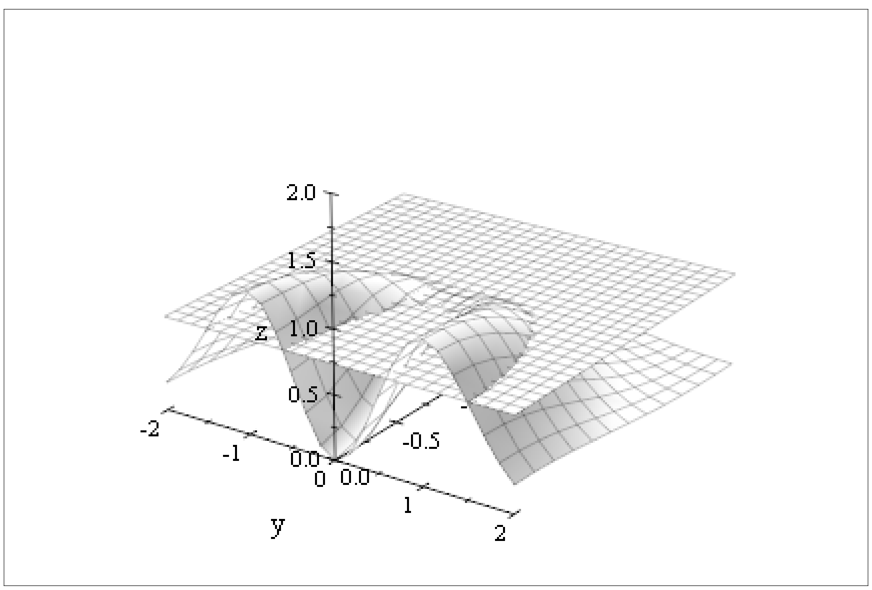
there is a saddle point at $\left(1, 0, 2e^{-1}\right)$





Similarly we can see that there is a saddle point at $\left(-1, 0, 2e^{-1}\right)$





For $(0, 1)$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,1)} = \left[4 - 20(0)^2 - 6(1)^2 + 8(0)^4 + 12(1)^2(0)^2\right]e^{-\left(0^2+1^2\right)} = -2e^{-1}$$

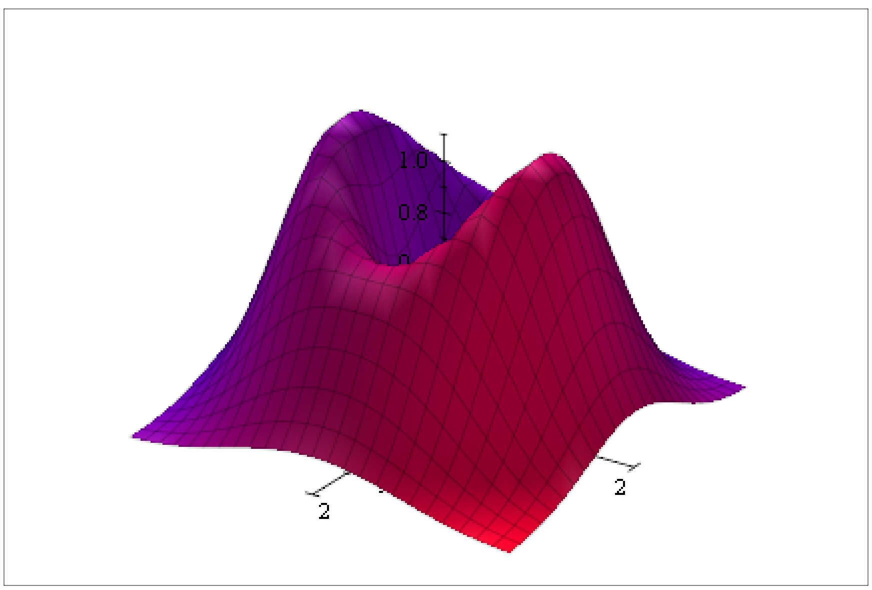
$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,1)} = \left[6 - 4(0)^2 - 30(1)^2 + 8(1)^2(0)^2 + 12(1)^4\right]e^{-\left(0^2+1^2\right)} = -24e^{-1}$$

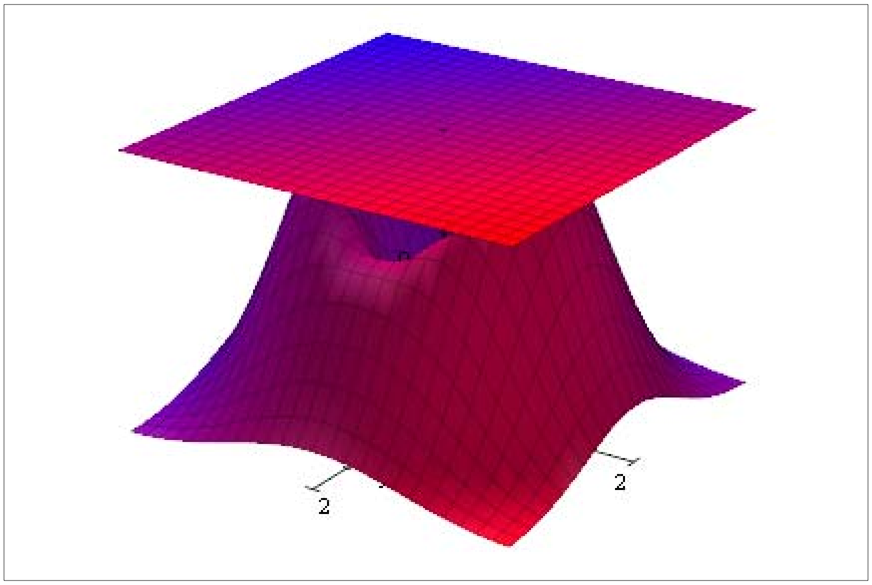
$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(1,0)} = \left[-20(0)(-1) + 8(0)^3(-1) + 12(0)(-1)^3\right]e^{-\left(0^2+1^2\right)} = 0$$

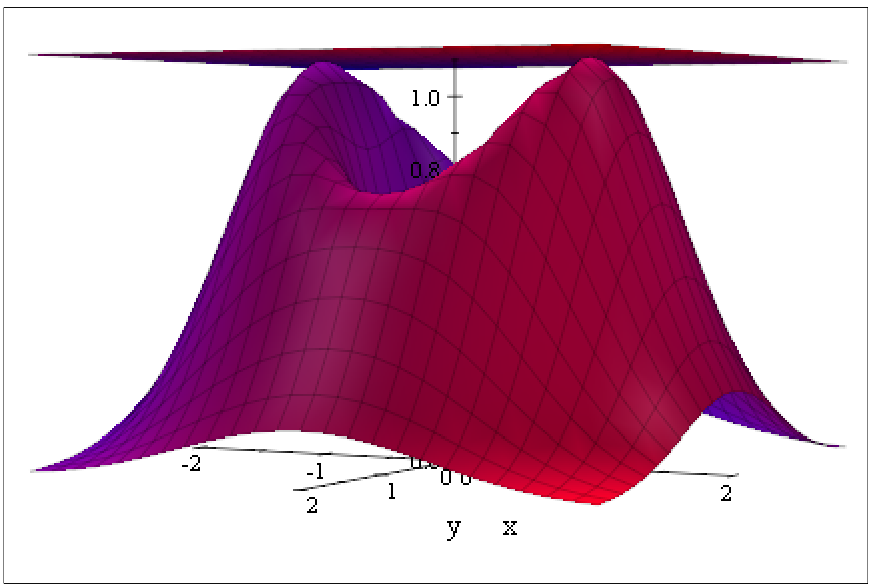
$$D_{(1,0)} = \left(-2e^{-1}\right)\left(-24e^{-1}\right) - 0^2 = 48e^{-2} > 0$$

$$f(0,1) = \left(2(0)^2 + 3(1)^2\right)e^{-\left(0^2+1^2\right)} = 3e^{-1}$$

there is a relative max $\left(0, 1, 3e^{-1}\right)$







Similarly, we can see that there is a relative Maximum at $\left(0,-1,3e^{-1}\right)$