Now that we know that

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ give the rates of change of the function $w = f(x,y,z)$ in along the x,y, and z axes respectively, we would like to have a way of obtaining the rate of change of a function in an arbitrary direction.

Let us first develop tools to do this.

Let $w = f(x,y,z)$ be a differentiable function

Define the gradient $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

that is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Example 1:

Let $f(x,y,z) = x^2 z \sin y$

$$\nabla f = \left\langle 2xz\sin y, x^2 z\cos y, x^2 \sin y \right\rangle$$

In particular at the point $\left(1, \frac{\pi}{2}, 1\right)$

$$\nabla f|_{\left(1,\frac{\pi}{2},1\right)}$$
$$= \left\langle 2(1)(1)\sin\left(\frac{\pi}{2}\right), (1)^2(1)\cos\left(\frac{\pi}{2}\right), (1)^2 \sin\left(\frac{\pi}{2}\right) \right\rangle$$
$$= \langle 2, 0, 1 \rangle$$

The directional derivative of the function $w = f(x,y,z)$

in the direction of the unit vector u is denoted and is given by

$$D_u(f(x,y,z)) = \nabla f(x,y,z) \cdot u$$

Example 2:

To find the directional derivative of $f(x,y,z) = x^2 z \sin y$ at $P\left(1, \frac{\pi}{4}, 2\right)$ in the direction of the unit vector $u = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$

Note that

$$\nabla f|_{\left(1,\frac{\pi}{4},2\right)}$$
$$= \left\langle 2(1)(2)\sin\left(\frac{\pi}{4}\right), (1)^2(2)\cos\left(\frac{\pi}{4}\right), (1)^2\sin\left(\frac{\pi}{4}\right) \right\rangle$$
$$= \left\langle 4 \times \frac{\sqrt{2}}{2}, 2 \times \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$
$$= \left\langle 2\sqrt{2}, \sqrt{2}, \frac{\sqrt{2}}{2} \right\rangle$$

therefore:

$$D_u(f(x,y,z))|_{\left(1,\frac{\pi}{4},2\right)}$$
$$= \left\langle 2\sqrt{2}, \sqrt{2}, \frac{\sqrt{2}}{2} \right\rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$$
$$= \frac{2\sqrt{2}}{3} + \frac{2\sqrt{2}}{3} - \frac{2\sqrt{3}}{2 \times 3}$$
$$= \frac{3\sqrt{2}}{3}$$
$$= 3$$

That is to say that the rate of change of the function f in the direction of u at $\left(1, \frac{\pi}{4}, 2\right)$ is 3 for each unit of change along u

Eaxmple 3:

To find the directional derivative of the function

$f(x,y,z) = e^{xyz}$ at $P(1,-1,0)$ in the direction of PQ with $Q(2,1,1)$

The unit vector u is a unit vector along $\overrightarrow{PQ}$

$$\overrightarrow{PQ} = \langle 2-1, 1-(-1), 1-0 \rangle = \langle 1,2,1 \rangle$$

therefore

$$\left|\left|\overrightarrow{PQ}\right|\right| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$u = \frac{\overrightarrow{PQ}}{\left|\left|\overrightarrow{PQ}\right|\right|} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$\nabla f = \langle yze^{xyz}, zxe^{xyz}, xye^{xyz} \rangle$$

$$\nabla f|_{(1,-1,0)} = \left\langle (-1)(0)e^0, (0)(1)e^0, (1)(-1)e^0 \right\rangle = \langle 0,0,-1 \rangle$$

Therefore:

$$D_u(f(x,y,z))|_{(1,-1,0)}$$
$$= \langle 0,0,-1 \rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$
$$= -\frac{1}{\sqrt{6}}$$
$$= -\frac{\sqrt{6}}{6}$$

Note that

$D_u(f(x,y,z)) = \nabla f(x,y,z) \cdot u$

means that

$D_u(f(x,y,z)) = \|\nabla f(x,y,z)\| \|u\| \cos\theta = \|\nabla f(x,y,z)\| \cos\theta$,
because $\|u\| = 1$

where θ is the angle between $\nabla f(x,y,z)$ and u, and $0 \le \theta \le \pi$,

We know that $\cos 0 = 1, \cos\pi = -1$ and $-1 \le \cos\theta \le 1$

i.e.

$-\|\nabla f(x,y,z)\| \le \|\nabla f(x,y,z)\| \cos\theta \le \|\nabla f(x,y,z)\|$

or

$-\|\nabla f(x,y,z)\| \le D_u(f(x,y,z)) \le \|\nabla f(x,y,z)\|$
therefore $D_u(f(x,y,z))$ has maximum value of $\|\nabla f(x,y,z)\|$
when $\nabla f(x,y,z)$ and u are in the same direction that is along the direction of $\nabla f(x,y,z)$

AND

$D_u(f(x,y,z))$ has minimum value of $\|\nabla f(x,y,z)\|$
when $\nabla f(x,y,z)$ and u are in the opposite direction that is along the direction of $-\nabla f(x,y,z)$

Example 4:

The temperature at a point (x, y, z) is given by
$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$

where T is measured in degrees C and x, y, z are in meters.

to find the direction, along which the temperature increases fastest at $P(2, -1, 2)$

and also to find the maximum rate of increase of temperature at
$P(2, -1, 2)$

The fastest increase is along $\nabla T|_{(2,-1,2)}$

$$\nabla T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle = \left\langle -400xe^{-x^2-3y^2-9z^2}, -1200ye^{-x^2-3y^2-9z^2}, -3600ze^{-x^2-3y^2-9z^2} \right\rangle = -400e^{-x^2-3y^2-9z^2}\langle x, 3y, 9z\rangle$$

$$\nabla T|_{(2,-1,2)} = -400e^{-2^2-3(-1)^2-9(2)^2}\langle 2, 3(-1), 9(2)\rangle = -400e^{-43}\langle 2, -3, 18\rangle$$

$$\left|\left|\nabla T|_{(2,-1,2)}\right|\right| = 400e^{-43}\sqrt{(2)^2 + (-3)^2 + 18^2}$$

$\left|\left|\nabla T|_{(2,-1,2)}\right|\right| = 400e^{-43}\sqrt{337}$ degrees C/meter is the greatest rate of change that occurs along the unit vector

$$\frac{\nabla T|_{(2,-1,2)}}{\left|\left|\nabla T|_{(2,-1,2)}\right|\right|}$$

You may practice with the exercises

1 thru 71 (odd numbered) pages 940-941