The Chain Rule:

Let $w = f(u,v)$

$u = g(x,y)$ **and** $v = h(x,y)$

and $f, g,$ **and** h **are differentiable.**

then

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

Example 1:

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To find $\frac{\partial w}{\partial s}$ **and** $\frac{\partial w}{\partial t}$ **at** $s = 0$ **and** $t = \frac{\pi}{2}$

For $w = \sin(2x + 3y)$ **,** $x = s + t$ **and** $y = s - t$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = \mathbf{2}\cos(2x + 3y)(1) + \mathbf{3}\cos(2x + 3y)(1) = \mathbf{5}\cos(2x + 3y)$$

For $s = 0$ **and** $t = \frac{\pi}{2}$**, we have** $x = \frac{\pi}{2}$ **and** $y = -\frac{\pi}{2}$

$\frac{\partial w}{\partial s}\Big|_{s=0,t=\frac{\pi}{2}} = \mathbf{5}\cos\left(2\left(\frac{\pi}{2}\right) - 3\left(\frac{\pi}{2}\right)\right) = \mathbf{5}\cos\left(-\frac{\pi}{2}\right) = \mathbf{0}$

$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = \mathbf{2}\cos(2x+3y)(1) + \mathbf{3}\cos(2x+3y)(-1) = -\cos(2x+3y)$

For $s = 0$ **and** $t = \frac{\pi}{2}$**, we have** $x = \frac{\pi}{2}$ **and** $y = -\frac{\pi}{2}$

$\frac{\partial w}{\partial t}\Big|_{s=0,t=\frac{\pi}{2}} = -\cos\left(2\left(\frac{\pi}{2}\right) + 3\left(-\frac{\pi}{2}\right)\right) = \mathbf{0}$

Example 2:

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To find $\frac{\partial w}{\partial s}$ **and** $\frac{\partial w}{\partial t}$

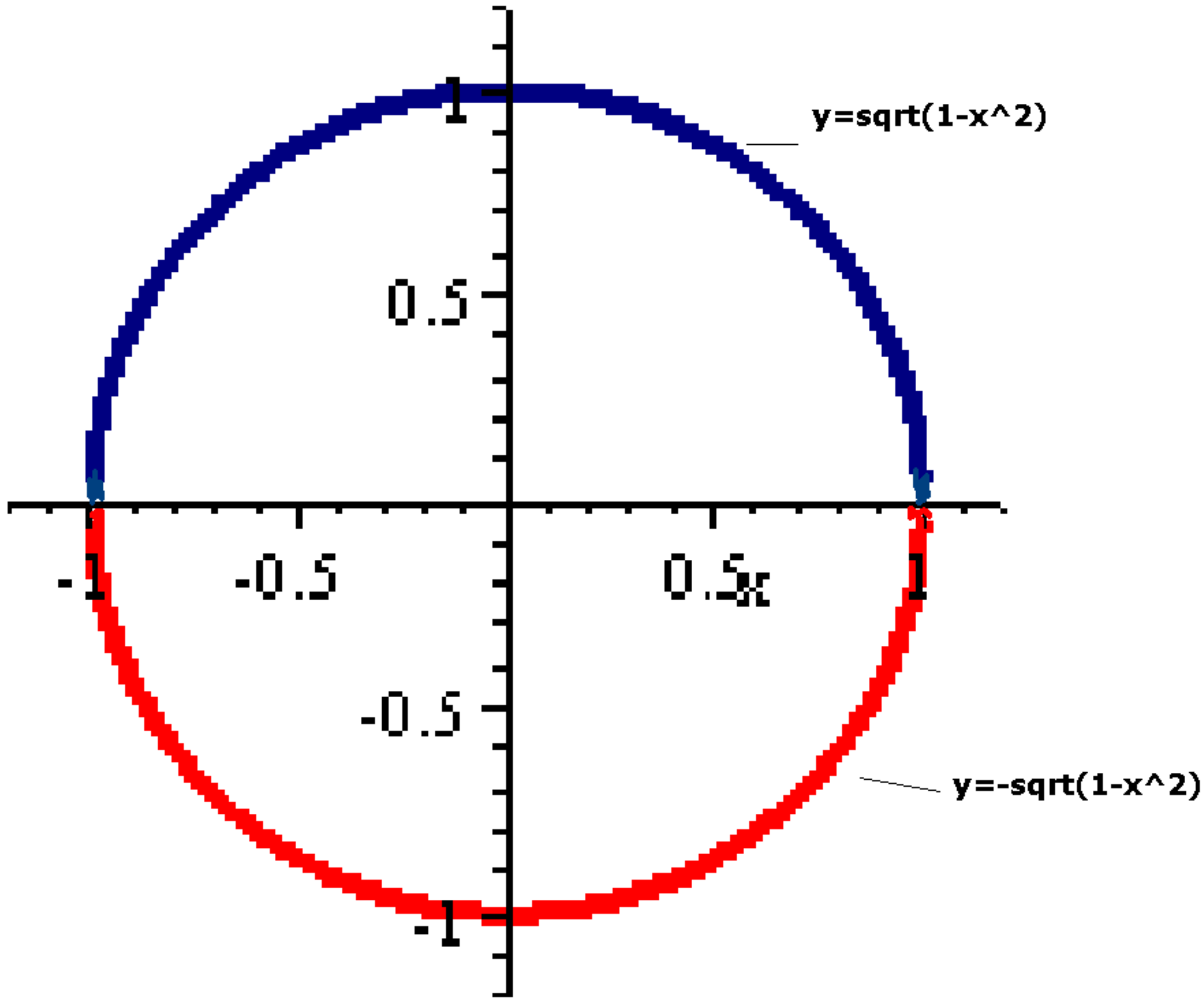
For $w = x\cos(yz)$ **,** $x = s^2$**,** $y = t^2$**, and** $z = s - 2t$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \cos(yz)(2s) - \mathbf{xz}\sin(yz)(0) - \mathbf{xy}\sin(yz)(1) = \mathbf{2s}\cos(yz) - \mathbf{xy}\sin(yz) = \mathbf{2s}\cos(t^2s - 2t^3) - \mathbf{s}^2\mathbf{t}^2\sin(t^2s - 2t^3)$

$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = \cos(yz)(0) - \mathbf{xz}\sin(yz)(2t) - \mathbf{xy}\sin(yz)(-2) = -\mathbf{2t}(s^3 - 2s^2t)\sin(t^2s - 2t^3) + \mathbf{2s}^2\mathbf{t}^2\sin(t^2s - 2t^3)$

Implicit differentiation using the partial derivatives

Recall that the equation $x^2 + y^2 - 1 = 0$ **gives** y **a s a function of** x **implicitly**



If we used, $F(x,y) = x^2 + y^2 - 1$

$F(x,y) = 0$ **whenever we have** $y = \sqrt{1-x^2}$ **or** $y = -\sqrt{1-x^2}$

Now that we have x **as a function of** x **and** y **as a function of** x

Differentiation of $F(x,y) = 0$

with respect to x **will give**

$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} =$ **0**

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} =$ **0**

or

$$\boxed{\frac{dy}{dx} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\right)}}$$

This rule applies in all the situations, where $F(x,y) = 0$ **gives** y **implicitly as a differentable function of** x

and we have the derivatives available alongwith $\left(\frac{\partial F}{\partial y}\right) \neq 0$

The same rule extends when we have

$F(x,y,z) = 0$ **gives** z **implicitly as a differentiable function of** x **and** y

then

$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)}$ **and** $\frac{\partial z}{\partial y} = -\frac{\left(\frac{\partial F}{\partial y}\right)}{\left(\frac{\partial F}{\partial y}\right)}$

Example 3:

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To find $\frac{\partial z}{\partial x}$ **and** $\frac{\partial z}{\partial y}$

where

$\mathbf{x} \ln \mathbf{y} + \mathbf{y}^2\mathbf{z} + \mathbf{z}^2 = \mathbf{8}$

take $F(x,y,z) = x\ln y + y^2z + z^2 - 8$

$$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{\ln y}{y^2+2z}$$

$$\frac{\partial z}{\partial y} = -\frac{\left(\frac{x}{y}+2yz\right)}{(y^2+2z)}$$

Example 4

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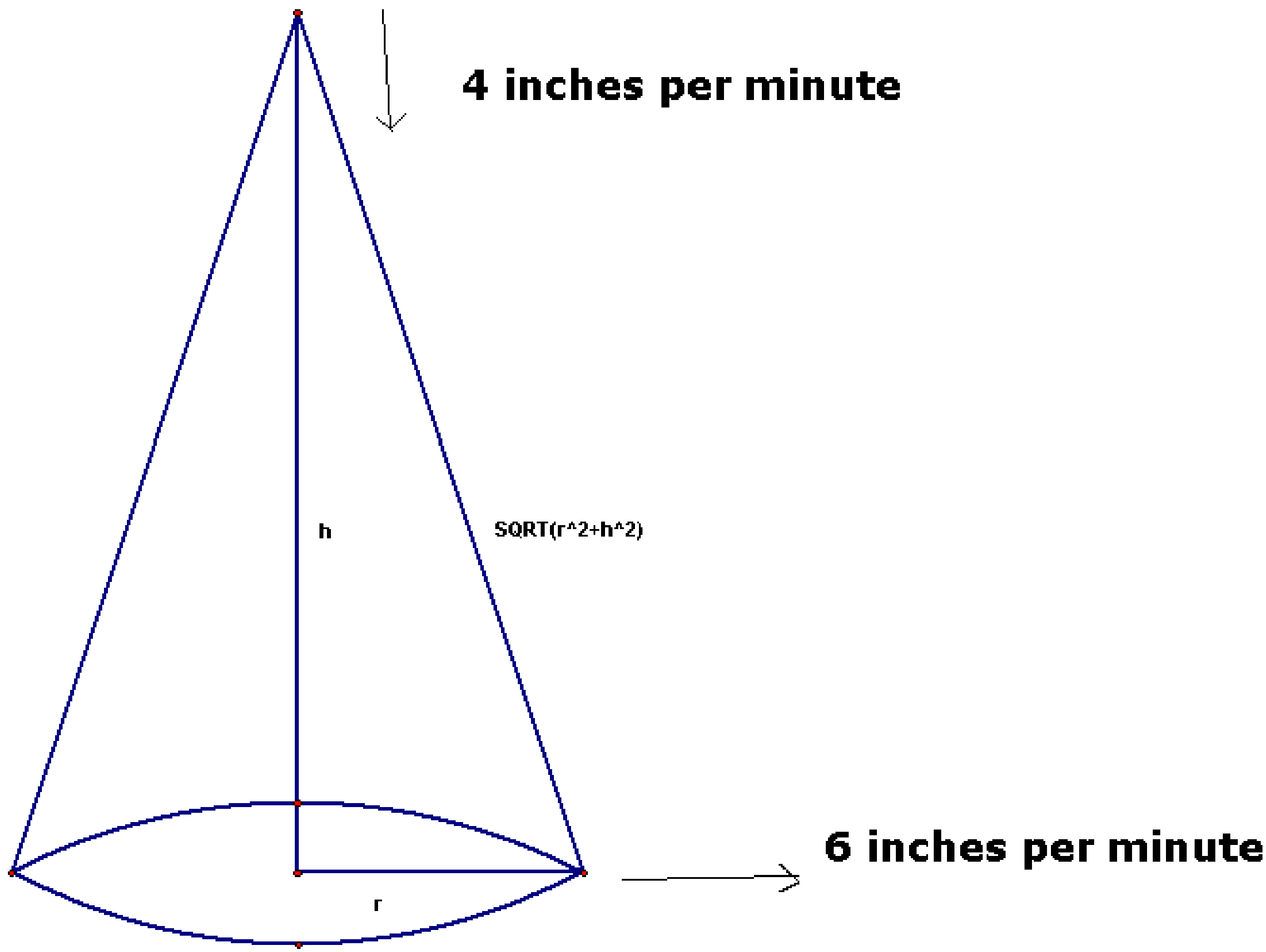
V: volume, S: surface area, t: time in minutes



$\frac{dr}{dt} = 6$ **inches per minute**

$\frac{dh}{dt} = -4$ **inches per minute**

The volume $V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$

$\frac{dV}{dt} = \frac{2}{3}\boldsymbol{\pi}\mathbf{rh}(6) + \frac{1}{3}\boldsymbol{\pi}\mathbf{r}^2(-4)$

or

$\frac{dV}{dt} = \mathbf{4}\boldsymbol{\pi}\mathbf{rh} - \frac{4}{3}\boldsymbol{\pi}\mathbf{r}^2$

$\frac{dV}{dt}\Big|_{r=12,h=36} = 4\pi(12)(36) - \frac{4}{3}\pi(12)^2 = 1536\pi \ \ in^3/\min$

The surface area $S = \pi r\sqrt{r^2 + h^2} + \pi r^2$ **(lateral area plus the area of the base)**

$\frac{dS}{dt} = \frac{\partial S}{\partial r}\frac{dr}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt}$

$\frac{dS}{dt} = \left(\pi\sqrt{r^2 + h^2} + \pi r\frac{r}{\sqrt{r^2+h^2}} + 2\pi r\right)(6) + \left(\frac{\pi rh}{\sqrt{r^2+h^2}}\right)(-4)$

$\frac{dS}{dt}\Big|_{r=12,h=36} = \left(\pi\sqrt{12^2 + 36^2} + \pi\frac{12^2}{\sqrt{12^2+36^2}} + 2\pi(12)\right)(6) + \left(\frac{\pi(12)(36)}{\sqrt{12^2+36^2}}\right)(-4)$

$\left(\pi\sqrt{12^2 + 36^2} + \pi\frac{12^2}{\sqrt{12^2+36^2}} + 2\pi(12)\right)(6) + \left(\frac{\pi(12)(36)}{\sqrt{12^2+36^2}}\right)(-4) = \frac{324}{5}\pi\sqrt{10} + 144\pi \ \ in^2/\min$

Suggested Practice

Section 15.5:

3,9,13,17,21,25,33,41,43,51