Let us extend the notion of differentials in the case of the functions of one variable to a multivariable function.

Consider the function $z = f(x,y)$

If we change x **to** $x + \Delta x$ **and** y **to** $y + \Delta y$

the change in the value of z **is**

$$\Delta \mathbf{z} = \mathbf{f}(\mathbf{x} + \Delta\mathbf{x}, \mathbf{y} + \Delta\mathbf{y}) - \mathbf{f}(x,y)$$

often, it is convenient to approximate Δz **with the total differential**

$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ **where** $dx = \Delta x$ **and** $\Delta y = dy$

For a function $w = f(x,y,z)$

$$\mathbf{dw} = \frac{\partial f}{\partial x}\mathbf{dx} + \frac{\partial f}{\partial y}\mathbf{dy} + \frac{\partial f}{\partial z}\mathbf{dz}$$

Example 1

#10 on the page 921

To find dw

if $w = x^2yz^2 + \sin(yz)$

$$\mathbf{dw} = \frac{\partial f}{\partial x}\mathbf{dx} + \frac{\partial f}{\partial y}\mathbf{dy} + \frac{\partial f}{\partial z}\mathbf{dz}$$

$=2xyz^2dx + (x^2z^2 + z\cos(yz))dy + (2x^2yz + y\cos(yz))dz$

Example 2:

#14 on the page 921

For the function $f(x,y) = xe^y$

take $z = f(x,y)$
a) Evaluate $f(1,2)$ **,** $f(1.05, 2.1)$ **and calculate** Δz

b) Finddz

$\mathbf{f}(\mathbf{1},\mathbf{2}) = \mathbf{e}^2 = \mathbf{7.3890560989306502274}$
$\mathbf{f}(1.05, 2.1) = \mathbf{1.05e}^{2.1} = \mathbf{8.5744784081960325774}$

$\Delta\mathbf{z} = \mathbf{f}(1.05, 2.1) - \mathbf{f}(\mathbf{1},\mathbf{2}) = \mathbf{8.5744784081960325774} - \mathbf{7.3890560989306502274} = \mathbf{1.18542230926538235}$

with $dx = .05, dy = .1$

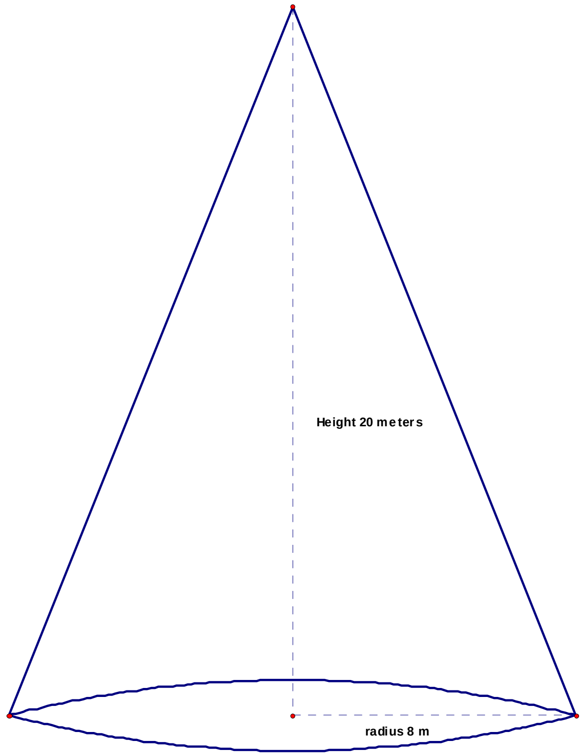
dz
$= \frac{\partial f}{dx}\mathbf{dx} + \frac{\partial f}{\partial y}\mathbf{dy}$
$= e^y dx + xe^y dy$

$\mathbf{dz} = \mathbf{e}^2(.05) + (1)\mathbf{e}^2(.1) = \mathbf{1.1083584148395975341}$

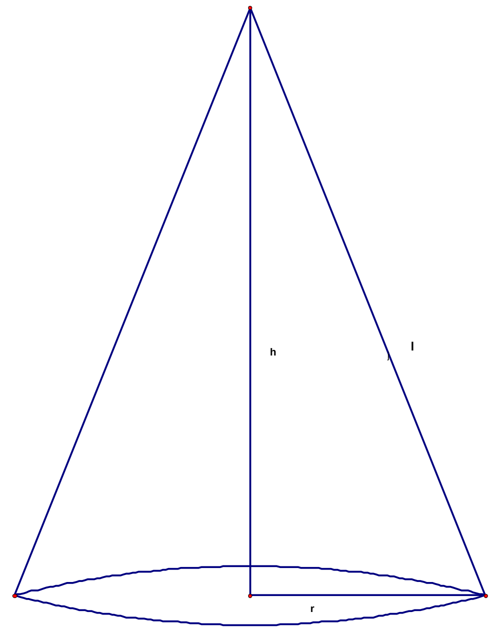
Example 3: Note that for small values of Δx **and** Δy **,** dz **is a very nice approximation of** Δz

#26 on the page 921

Given that



Remember that the lateral surface area of the cone is $\pi r l$



l $=\sqrt{r^2+h^2}$

The lateral surface area is $S=\pi r\sqrt{r^2+h^2}$

or

$\mathbf{S}(r,h)=\boldsymbol{\pi}\mathbf{r}\sqrt{r^2+h^2}$

therefore the total differential is

$\mathbf{dS}=\frac{\partial S}{\partial r}\mathbf{dr}+\frac{\partial S}{\partial h}\mathbf{dh}$

$$\frac{\partial S}{\partial r} = \boldsymbol{\pi}\sqrt{r^2+h^2} + \boldsymbol{\pi}\mathbf{r}\frac{r}{\sqrt{r^2+h^2}} = \boldsymbol{\pi}\left(\frac{r^2+h^2+r^2}{\sqrt{r^2+h^2}}\right) = \boldsymbol{\pi}\frac{2r^2+h^2}{\sqrt{r^2+h^2}}$$

$$\frac{\partial S}{\partial h} = \frac{\pi r h}{\sqrt{r^2+h^2}}$$

$$\mathbf{dS} = \left(\pi\frac{2r^2+h^2}{\sqrt{r^2+h^2}}\right)\mathbf{dr} + \left(\frac{\pi r h}{\sqrt{r^2+h^2}}\right)\mathbf{dh}$$

$r = 8m \qquad h = 20m$

$\mathbf{S}(8,20) = \boldsymbol{\pi}(8)\sqrt{20^2+8^2}$

Δr	Δh	dS	ΔS
$.1$	$.1$	$\left(\pi\frac{2(8)^2+(20)^2}{\sqrt{8^2+20^2}}\right)(.1) + \left(\frac{\pi(8)(20)}{\sqrt{8^2+20^2}}\right)(.1) \cong 10.03412$	$\pi(8.1)\sqrt{20.1^2+8.1^2} - \pi(8)\sqrt{20^2+8^2} \cong 10.07678$
$.1$	$-.1$	$\left(\pi\frac{2(8)^2+(20)^2}{\sqrt{8^2+20^2}}\right)(.1) + \left(\frac{\pi(8)(20)}{\sqrt{8^2+20^2}}\right)(-.1) \cong 5.36709$	$\pi(8.1)\sqrt{19.9^2+8.1^2} - \pi(8)\sqrt{20^2+8^2} \cong 5.35959$
$.001$	$.002$	$\left(\pi\frac{2(8)^2+(20)^2}{\sqrt{8^2+20^2}}\right)(.001) + \left(\frac{\pi(8)(20)}{\sqrt{8^2+20^2}}\right)(.002) \cong 0.12368$	$\pi(8.001)\sqrt{20.002^2+8.001^2} - \pi(8)\sqrt{20^2+8^2} \cong 0.12368$
$-.0001$	$.0002$	$\left(\pi\frac{2(8)^2+(20)^2}{\sqrt{8^2+20^2}}\right)(-.0001) + \left(\frac{\pi(8)(20)}{\sqrt{8^2+20^2}}\right)(.0002) \cong -3.03\times 10^{-3}$	$\pi(7.9999)\sqrt{20.0002^2+7.9999^2} - \pi(8)\sqrt{20^2+8^2} \cong -0.0030$

Example 4:

#46 on the page 922

For the function

$$\mathbf{f}(x,y)=\begin{cases} \frac{5x^2y}{x^3+y^3} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

First, to find $\frac{\partial f}{\partial x}\Big|_{(0,0)}$ **and** $\frac{\partial f}{\partial y}\Big|_{(0,0)}$

$$f_x(0,0) = \frac{\partial f}{\partial x}\Big|_{(0,0)} = Lim_{\Delta x \to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = Lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$f_y(0,0) = \frac{\partial f}{\partial y}\Big|_{(0,0)} = Lim_{\Delta y \to 0} \frac{f(0,\Delta y)-f(0,0)}{\Delta y} = Lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

therefore both the partial derivatives exist at (0.0)

To check the differentiability at $(0,0)$

Since continuity at $(0,0)$ **is a necessary condition for diffentiability at** $(0,0)$

let us first check for continuity at $(0,0)$

For the continuity, we need to have $Lim_{(x,y)\to(0,0)} f(x,y)$

For this function, if we approach $(0,0)$ **via a line** $y = x$ **, the value of f approaches** $\frac{5x^2(x)}{x^3+x^3} = \frac{5}{2}$

if we approach $(0,0)$ **via a line** $y = 2x$ **, the value of f approaches** $\frac{5x^2(2x)}{x^3+(2x)^3} = \frac{10x^3}{9x^3} = \frac{10}{9}$

The function approaches different values if we approach $(0,0)$ **along different paths, therefore the limit does not exist at** $(0,0)$ **and the function is not continuous and consequently not differentiable at** $(0,0)$

Suggested Practice:

Section 13.4: 1 thru 27 odd, 31, 35, 45