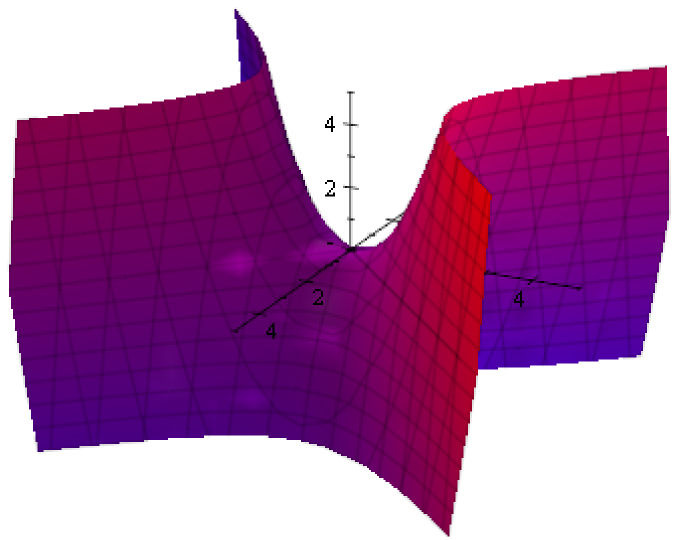
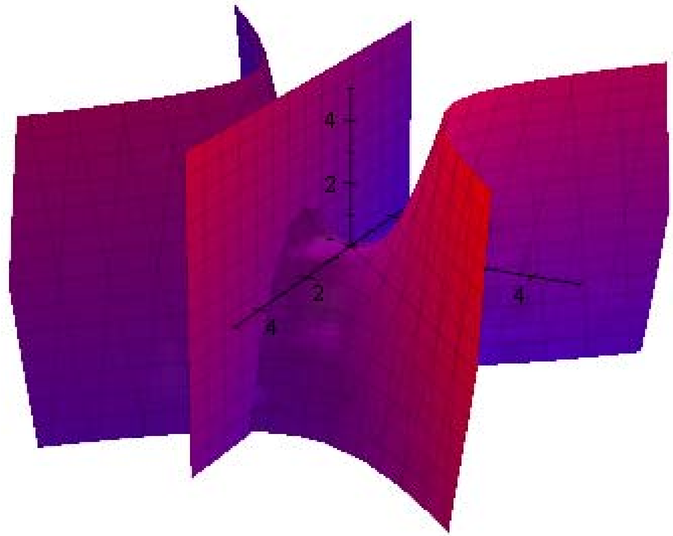
Partial Derivatives:

Consider the function: $f(x,y) = y^2 - x^2$



We would like to see the rate of change of this function, when we keep one of the two independent variables as a constant.

For example, in the above function, if we set y **as a constant, the graph will be the curve formed by the intersection of the two surfaces shown below**



To find the rate of change of this function along this particlar curve, we shall treat the y-value as that particular constant and look at the rate of change in wrt x

that may be performed by differentiating $f(x,y)$ wrt x treating y as a constant.

in case, the following limit is available:

Lim$_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{x}$

the notation for this limit is $\frac{\partial f}{\partial x}$

Example 1: Find $\frac{\partial f}{\partial x}$ **and** $\frac{\partial f}{\partial y}$ **if** $f(x,y) = x\sin y^2$

When we compute $\frac{\partial f}{\partial x}$, we treat y as a constant therefore $\sin y^2$ as a constant and $\frac{\partial f}{\partial x} = \sin y^2$

For $\frac{\partial f}{\partial y}$, we treat x as a constant, therefore

$\frac{\partial f}{\partial y} = x\,\frac{\partial}{\partial y}(\sin y^2) = x(2y\cos y^2)$ **, Note that we used the chain rule and also that $\frac{\partial}{\partial y}(\sin y^2)$ has the same value as $\frac{d}{dy}(\sin y^2)$ does.**

Example 2:

Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ if it is given that $f(x,y,z) = x^{y^3 z}$

For the computation of $\frac{\partial f}{\partial x}$, note that y and z will be treated as constants therefore $y^3 z$ is a constant and we are differentiating x to a constant power.

Therefore the calculation is fairly simple,

$\frac{\partial f}{\partial x} = (y^3 z)x^{y^3 z-1}$ $\boxed{\text{Because, } \frac{dx^n}{dx} = nx^{n-1} \text{, in this case the constant } n = y^3 z}$

For the computation of $\frac{\partial f}{\partial y}$, note that x and z will be treated as constants

Now, we have a different procedure, in the case of $\frac{\partial f}{\partial x}$ for $f(x,y,z) = x^{y^3 z}$, we had a variable base and a constant power. For $\frac{\partial f}{\partial y}$ with $f(x,y,z) = x^{y^3 z}$ we have a constant base and a variable power.

Remember that $\boxed{\frac{da^u}{dt} = a^u(\ln a)\frac{du}{dt} \text{ for a constant } a > 0}$

$\frac{\partial}{\partial y}\left(x^{y^3 z}\right) = \mathbf{x}^{y^3 z}(\ln x)\frac{\partial}{\partial y}(y^3 z) = \mathbf{x}^{y^3 z}(\ln x)(3y^2 z) = \mathbf{3y^2 z}\left(x^{y^3 z}\right)\ln \mathbf{x}$

....

Higher order partial derivatives follow the very much the same style that the derivatives of the functions of a single variable do. But, we shall keep noticing the additional requirements as well as assumptions.

Example 3:

To find $\frac{\partial^2 f}{\partial x^2}$

for $f(x,y,z) = \sin(x^3 y e^z)$

$\frac{\partial f}{\partial x} = 3x^2 y e^z \cos(x^3 y e^z)$ **used the Chain Rule for differentiation wrt** x **, treated** ye^z **as a constant**

For

$\frac{\partial f}{\partial x}$

we shall differentiate

$\frac{\partial f}{\partial x} = 3x^2 y e^z \cos(x^3 y e^z)$ **with respect to** x

we shall use the product rule combined with the chain rule and ye^z **is treated as a contstant**

$\frac{\partial^2 f}{\partial x^2}$

$= \mathbf{3}(2x)\mathbf{ye}^z \cos(x^3ye^z)+\mathbf{3x}^2\mathbf{ye}^z(-3x^2ye^z\sin(x^3ye^z))$

$= \mathbf{6xye}^z\cos(x^3ye^z)-\mathbf{9x}^4\mathbf{y}^2\mathbf{e}^{2z}\sin(x^3ye^z)$

Example 4:

To evaluate $\frac{\partial^2 f}{\partial x \partial y}$ **for** $f(x,y) = xe^y$

$\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}(xe^y)= \mathbf{e}^y$

A partial derivative like $\frac{\partial^2 f}{\partial x \partial y}$ **is called a MIXED PARTIAL DERIVATIVE**

In case both $\frac{\partial^2 f}{\partial x \partial y}$ **and** $\frac{\partial^2 f}{\partial y \partial x}$ **are continous on an open disk D,**

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ **at each point in D.**

Example 5:

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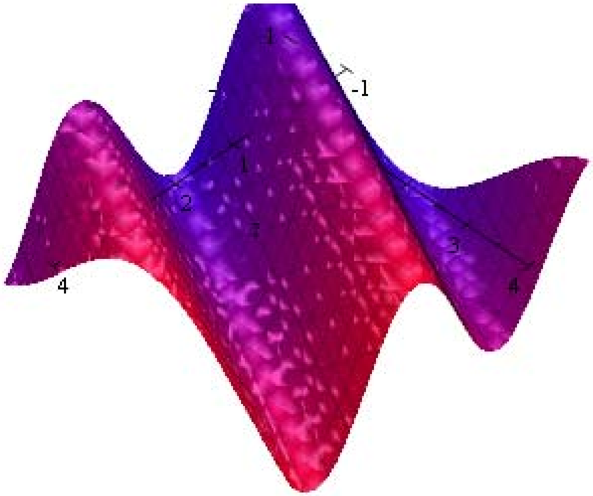
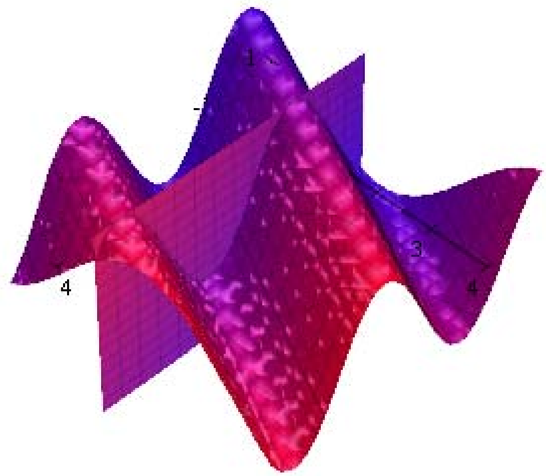
For the function $z = \cos(2x - y)$



To find the slopes of the surface in the x-direction and in the y-direction at the point $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

The slope in the x-driection at $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ **is the slope of the tangent line to the curve that is formed by taking the intersection of the surface** $z = \cos(2x - y)$ **with the plane** $y = \frac{\pi}{3}$



The slope is obtained by evaluating $\frac{\partial z}{\partial x}$ **at** $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

For $z = \cos(2x - y)$

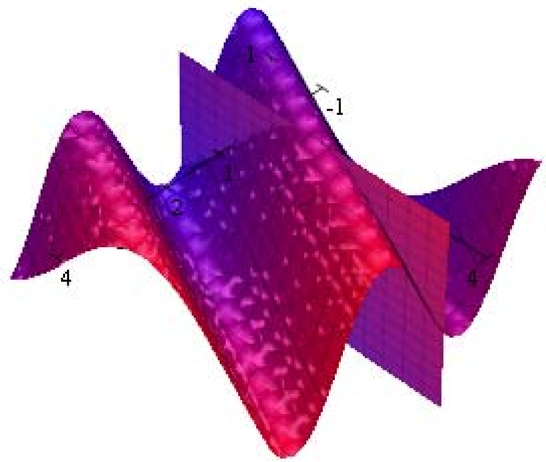
$\frac{\partial z}{\partial x} = -\mathbf{2}\sin(2x - y)$

at $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

$\frac{\partial z}{\partial x}\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)}$ **is a notation for the value of** $\frac{\partial z}{\partial x}$ **at** $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

$\frac{\partial z}{\partial x}\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)} = -2\sin\left(2\left(\frac{\pi}{4}\right) - \frac{\pi}{3}\right) = -1$

The slope in the y-driection at $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ **is the slope of the tangent line to the curve that is formed by taking the intersection of the surface** $z = \cos(2x - y)$ **with the plane** $x = \frac{\pi}{4}$



z $=\cos(2x-y) \rightarrow \frac{\partial z}{\partial y} = -(-1)\sin(2x-y)$

$\frac{\partial z}{\partial y}\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)} = \sin\left(2\left(\frac{\pi}{4}\right) - \frac{\pi}{3}\right) = \frac{1}{2}$

Example 5:

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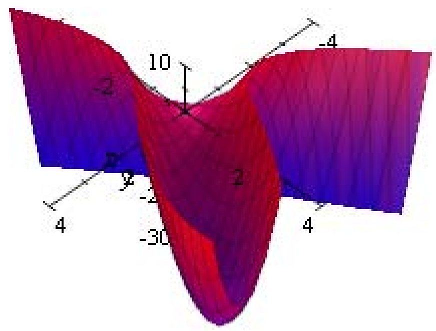
For the surface given by $f(x,y) = 3x^3 - 12xy + y^3$



to find the coordinates of all the points at which

$\frac{\partial f}{\partial x} = 0$ **and** $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = \mathbf{9x}^2 \mathbf{-12y}$

$\frac{\partial f}{\partial y} = \mathbf{-12x} + \mathbf{3y}^2$

$\frac{\partial f}{\partial x} = 0$ **and** $\frac{\partial f}{\partial y} = 0$

implies that

$9x^2 - 12y = 0$ **and** $-12x + 3y^2 = 0$

$\mathbf{9x}^2 \mathbf{-12y} = \mathbf{0} \rightarrow \mathbf{12y} = \mathbf{9x}^2 \rightarrow \mathbf{y} = \frac{9x^2}{12}$

substitute $y = \frac{3x^2}{4}$

in $-12x + 3y^2 = 0$

to obtain $-12x + 3\left(\frac{3x^2}{4}\right)^2 = 0$

which gives
$\mathbf{-12x} + \frac{27x^4}{16} = \mathbf{0}$
$\rightarrow$
$\mathbf{-4x} + \frac{9}{16}\mathbf{x}^4 = \mathbf{0}$
$\rightarrow$
$\mathbf{x}\left(\frac{9}{16}x^3 - 4\right) = \mathbf{0}$
$\rightarrow$

$x = 0$ **or** $\frac{9}{16}x^3 - 4 = 0 \rightarrow 9x^3 = 64 \rightarrow x^3 = \frac{64}{9} \rightarrow x = \frac{4}{9^{1/3}} \rightarrow x = \frac{4}{3^{2/3}}$

substitute these values in

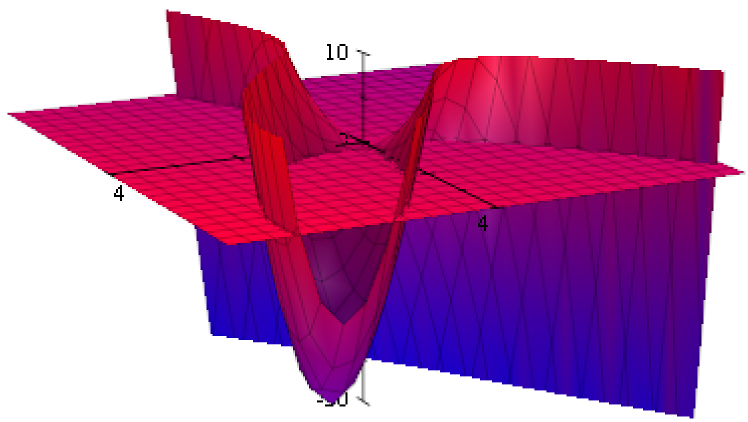
$\mathbf{9x}^2 \mathbf{-12y} = \mathbf{0}$

$\mathbf{x} = \mathbf{0} \rightarrow \mathbf{y} = \mathbf{0}$

$x = \frac{4}{3^{2/3}} \to 9\left(\frac{4}{3^{2/3}}\right)^2 - 12y = 0 \to 3\left(\frac{16}{3^{4/3}}\right) - 4y = 0 \to \frac{16}{3^{1/3}} - 4y = 0 \to y = \frac{4}{3^{1/3}}$

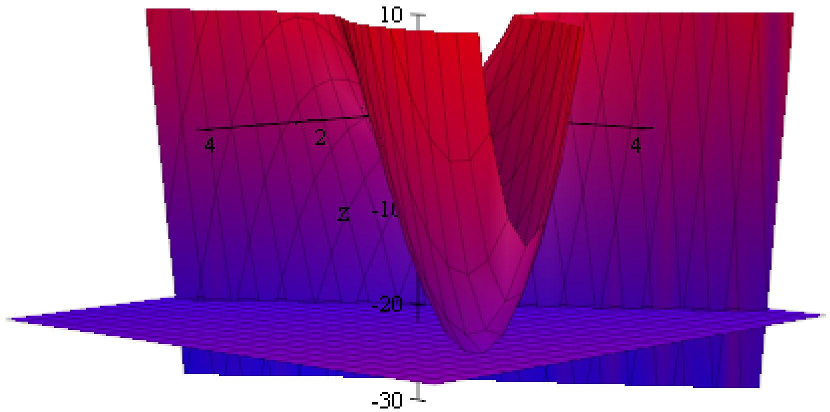
The coordinates of the required points are $(0,0)$ **and** $\left(\frac{16}{3^{1/3}}, \frac{4}{3^{1/3}}\right)$

at $(0,0)$ **we have**



The tangent plane is parallel to the xy-plane

at $\left(\frac{16}{3^{1/3}}, \frac{4}{3^{1/3}}\right)$



again the tangent plane is parallel to the xy-plane

Suggested Practice:

Section 13.3:

1 thru 97 odd numbered, 107 (must do)