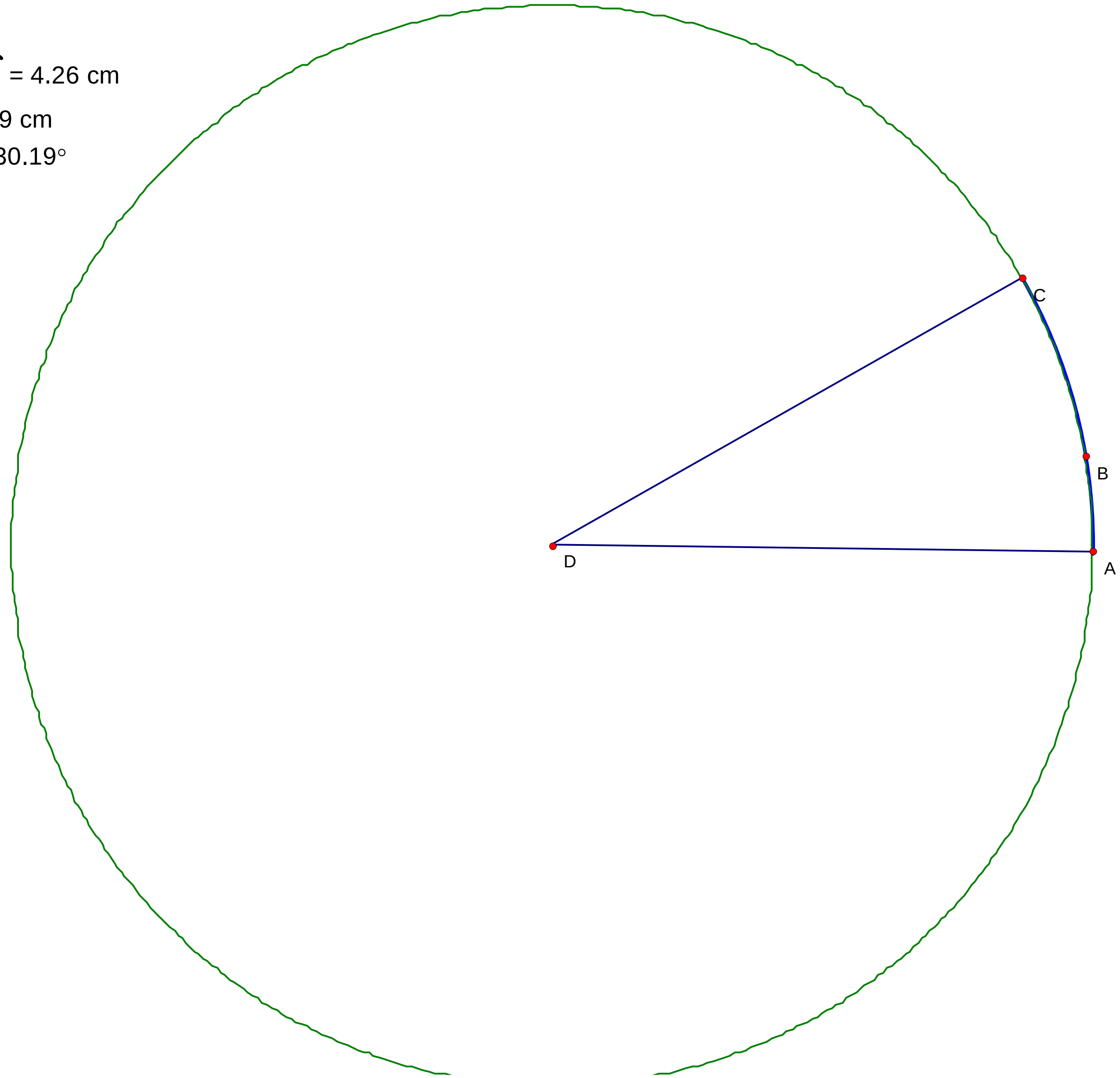
Lesson4 Part2

This lesson corresponds to the section 12.5 of your text book.

Length $\overset{\frown}{ABC}$ = 4.26 cm

m $\overline{DA}$ = 8.09 cm

m∠ADC = 30.19°



Let us recall that the length of the arc ABC shown in the above picture is

the radius of the circle times the measure of the angle ADC (in radians)

In the above picture

The arc ABC is $8.09 \times (30.19) \times \left(\frac{\pi}{180}\right) = 4.2627415505226427232$ **cm**

Remember the approach that you took in Calculus 2 for curves in a plane

For a smooth parametric curve that does intersect inteslf (except possibly for the end points of the interval under question)

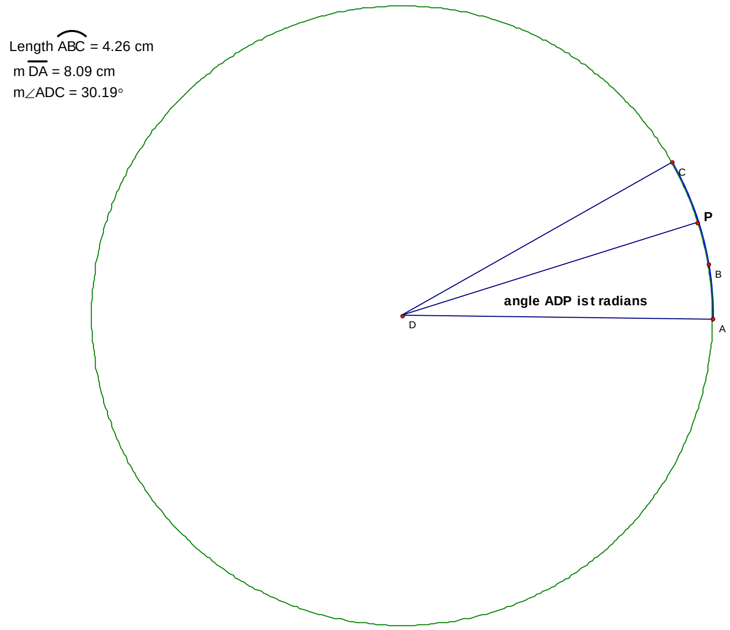
$\mathbf{x} = \mathbf{f}(\mathbf{t}) \qquad \mathbf{y} = \mathbf{g}(\mathbf{t}) \qquad \mathbf{a} \leq \mathbf{t} \leq \mathbf{b}$

The arc length from $t = a$ **to** $t = b$ **is given by** $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

For the circle shown above

we may write a parametric equation as $x = 8.09 \cos t \qquad y = 8.09 \sin t$

where t **is the angle ADP for a point P on the circumference of a circle,**



The arc length from $t = 0$ **to** $t = 30.19 \times \frac{\pi}{180} \cong 0.5269$ **radians is**

$$\int_0^{.5269} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\mathbf{dt}$$

$$= \int_0^{.5269} \sqrt{(-8.09\sin t)^2 + (8.09\cos t)^2}\,dt$$

$$= \int_0^{.5269} \sqrt{\left(8.09^2\sin^2 t + 8.09^2\cos^2 t\right)}\,dt$$

$$= \int_0^{.5269} \sqrt{\left(8.09^2 \sin^2 t + 8.09^2 \cos^2 t\right)}\, dt$$

$$= \int_0^{.5269} \sqrt{8.09^2 \left(\sin^2 t + \cos^2 t\right)}\, dt$$

$$= \int_0^{.5269} 8.09 dt$$

$= 8.09 \times .5269 = 4.262621$ **cm**

....

Just as we did the arc length problems in a parametrized plane curve,

for a smooth space curve traced by

$\vec{r}(t) =$ **x**(t)**i** $+$ **y**(t)**j** $+$ **z**(t)**k**

the arc length from $t = a$ **to** $t = b$

is given by

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, \mathbf{dt}$$

For an illustration, let us take up

exercise #12 on the page 875 of the textbook

To find the arc length from $t = 0$ to $t = \pi$ for

$\vec{r}(t) = \mathbf{2}\sin(t)\mathbf{i} + \mathbf{5t}\mathbf{j} + \mathbf{2}\cos(t)\mathbf{k}$

The required arc length is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,\mathbf{dt}$$

$$= \int_0^\pi \sqrt{(2\cos t)^2 + (5)^2 + (-2\sin t)^2}\,dt$$

$$= \int_0^\pi \sqrt{2^2\cos^2 t + (5)^2 + 2^2\sin^2 t}\,dt$$

$$= \int_0^\pi \sqrt{2^2\cos^2 t + 2^2\sin^2 t + 25}\,dt$$

$$= \int_0^\pi \sqrt{4\left(\cos^2 t + \sin^2 t\right) + 25}\,dt$$

$$= \int_0^\pi \sqrt{4 + 25}\,dt$$

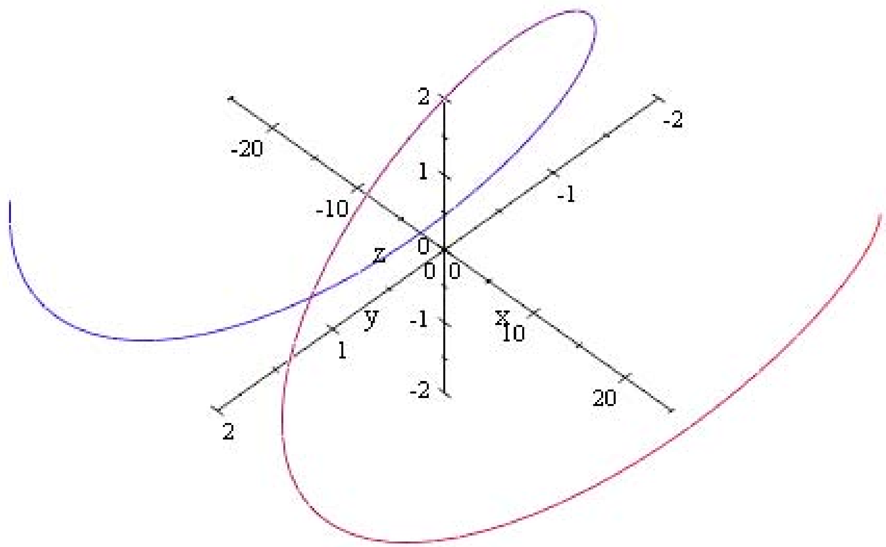
$$= \int_0^\pi \sqrt{29}\,dt$$

$$= \sqrt{29}\int_0^\pi dt$$

$$= \sqrt{29}\,\pi$$

$=\pi\sqrt{29}$

Remeber that such a curve is a helix

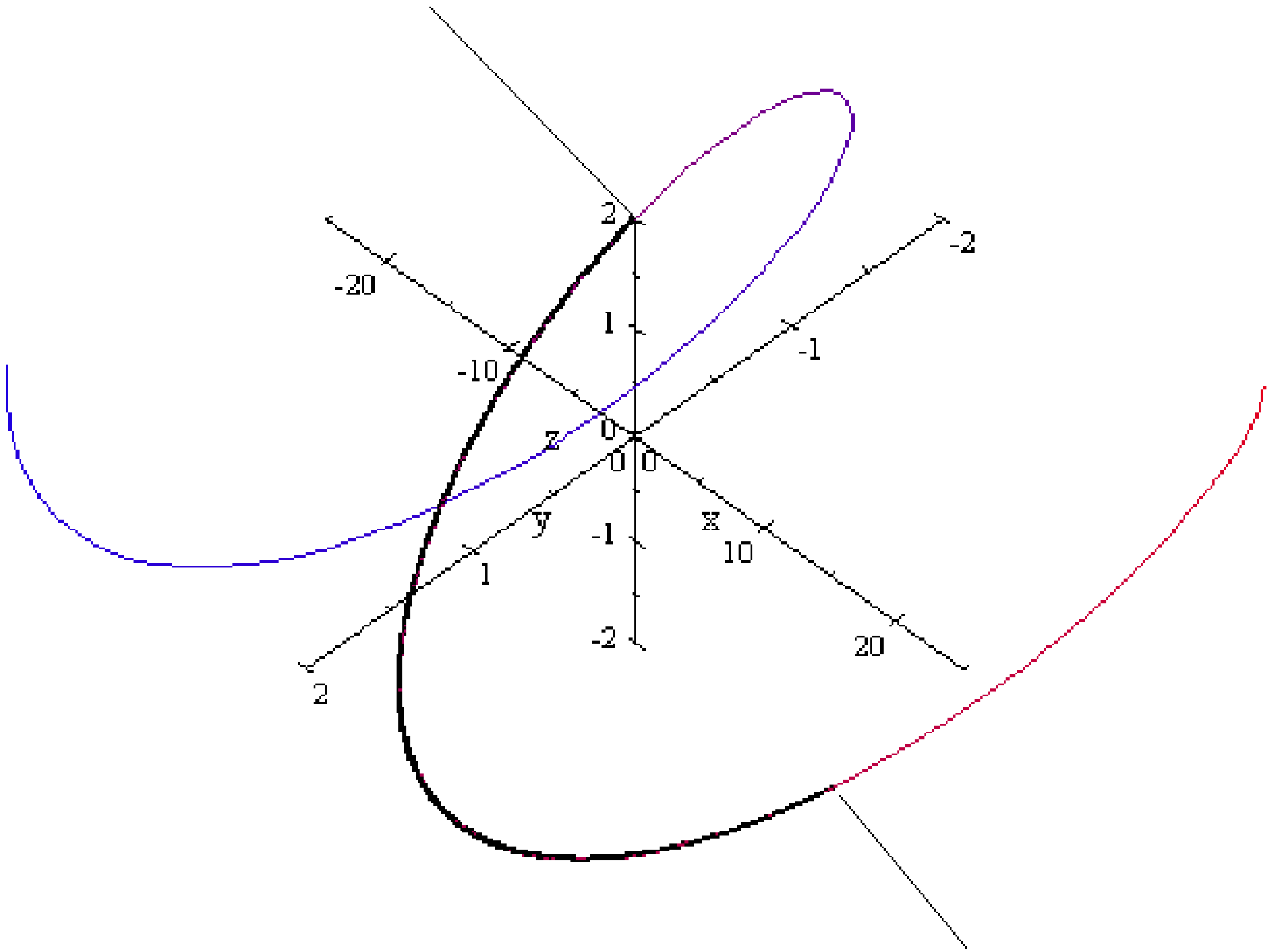


and we are looking for the arc length from the point with coordinates $(2\sin(0), 5(0), 2\cos(0)) = (0, 0, 2)$

and $(2\sin(\pi), 5(\pi), 2\cos(\pi)) = (0, 5\pi, -2)$

the darkened part has the arc length $\pi\sqrt{29}$

(0,0,2)



For a smooth curve that is given by $\vec{r}(t)$ on an interval $[a,b]$ we may define the arc length as a function of t by

$$s(t) = \int_a^t \left|\left|\vec{r}\,'(u)\right|\right| du \quad \textbf{OR} \quad s(t) = \int_a^t \sqrt{\left((x'(u))^2 + (y'(u))^2 + (z'(u))^2\right)}\, du$$

If we use The fundamental Theorem of Calculus**, we find that**

$$\frac{ds}{dt} = \sqrt{\left((x'(t))^2 + (y'(t))^2 + (z'(t))^2\right)} \quad \textbf{which equals } \left|\left|\frac{dr}{dt}\right|\right|$$

therefore, we get the following important fact that

$$\boxed{\frac{ds}{dt} = \left|\left|\frac{d\vec{r}}{dt}\right|\right|}$$

ALSO

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} \qquad \boxed{\text{Chain Rule}}$$

$$\left|\left|\frac{d\vec{r}}{dt}\right|\right| = \left|\left|\frac{d\vec{r}}{ds}\right|\right| \frac{ds}{dt}$$

using the fact that $\boxed{\frac{ds}{dt} = \left|\left|\frac{d\vec{r}}{dt}\right|\right|}$

we obtain

$\frac{ds}{dt} = \left|\left| \frac{d\vec{r}}{ds} \right|\right| \frac{ds}{dt}$

which gives us another important property that

$$\boxed{\left|\left| \frac{d\vec{r}}{ds} \right|\right| = 1}$$

Important Topic:

Curvature:

Curvature is a measure of the sharpness of the bending of a curve.

For smooth curve given by $\vec{r}(s) = \langle x(s), y(s), z(s) \rangle$ **where** s **is the arc length parameter, the curvature** K **at a point corresponding to** s **is given by** $K = \left|\left| \frac{dT}{ds} \right|\right| = \|T'(s)\|$

Look at the example #4 on the page 870 of the text to check that the curvature of a circle is the reciprocal of the radius.

Illustration:
#24 on the page 876
To find the curvature K **of the curve given by** $\vec{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \frac{3}{2}t^2 \right\rangle$

First, we have to change t **in terms of** s **to be able to use the definition**

K $= \left|\left| \frac{dT}{ds} \right|\right| = \|T'(s)\|$

Recall from the above that

$$s = \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du \qquad \textbf{for } \vec{r}(u) = \left\langle 4(\sin u - u\cos u), 4(\cos u + u\sin u), \tfrac{3}{2}u^2 \right\rangle$$

$$\mathbf{s} = \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,\mathbf{du}$$

$$\int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,\mathbf{du}$$

$$= \int_0^t \sqrt{(4(\cos u + u\sin u - \cos u))^2 + (4(-\sin u + u\cos u + \sin u))^2 + (3u)^2}\,\mathbf{du}$$

$$= \int_0^t \sqrt{(4u\sin u)^2 + (4u\cos u)^2 + (3u)^2}\,du$$

$$= \int_0^t \sqrt{16u^2\sin^2 u + 16u^2\cos^2 u + 9u^2}\,du$$

$$= \int_0^t \sqrt{16u^2\left(\sin^2 u + \cos^2 u\right) + 9u^2}\,du$$

$$= \int_0^t \sqrt{16u^2 + 9u^2}\,du$$

$=$

$\int_0^t \sqrt{25u^2}$ **du**

$=$

$\int_0^t$ **5udu**

$=$

$\frac{5t^2}{2}$

s $=\frac{5t^2}{2}$ → **t** $=\sqrt{\frac{2s}{5}}$

sustitute this value of t **in** $\vec{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \frac{3}{2}t^2 \right\rangle$

$\vec{r}(s)=\left\langle 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right), 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right), \frac{3}{5}s\right\rangle$

T(s)

$=\frac{d\vec{r}}{ds}$

$=\left\langle 4\left(\cos\sqrt{\frac{2s}{5}}\left(\sqrt{\frac{2s}{5}}\right)' - \left(\sqrt{\frac{2s}{5}}\right)'\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\left(\sqrt{\frac{2s}{5}}\right)'\right), 4\left(-\sin\sqrt{\frac{2s}{5}}\left(\sqrt{\frac{2s}{5}}\right)' + \left(\sqrt{\frac{2s}{5}}\right)'\sin\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\left(\right.\right.\right.$

$=\left\langle 4\left(\sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\left(\sqrt{\frac{2s}{5}}\right)'\right), 4\left(\sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\left(\sqrt{\frac{2s}{5}}\right)'\right), \frac{3}{5}\right\rangle$

$\left(\sqrt{\frac{2s}{5}}\right)' = \frac{d}{ds}\left(\sqrt{\frac{2s}{5}}\right) = \frac{1}{2}\left(\frac{2s}{5}\right)^{-1/2}\left(\frac{2}{5}\right) = \frac{1}{2}\left(\frac{5}{2s}\right)^{1/2}\left(\frac{2}{5}\right) = \frac{1}{\sqrt{10s}}$ **Chain Rule**

which gives

$$\sqrt{\frac{2s}{5}}\left(\sqrt{\frac{2s}{5}}\right)' = \sqrt{\frac{2s}{5}} \bullet \frac{1}{\sqrt{10s}} = \frac{1}{5}$$

therefore

$$T(s) = \left\langle 4\left(\sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\left(\sqrt{\frac{2s}{5}}\right)'\right), 4\left(\sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\left(\sqrt{\frac{2s}{5}}\right)'\right), \frac{3}{5}\right\rangle = \left\langle \frac{4}{5}\sin\sqrt{\frac{2s}{5}}, \frac{4}{5}\cos\sqrt{\frac{2s}{5}} . \frac{3}{5}\right\rangle$$

$$\mathbf{T}'(s) = \left\langle \frac{4}{5} \cdot \frac{1}{\sqrt{10s}}\cos\sqrt{\frac{2s}{5}}, -\frac{4}{5} \cdot \frac{1}{\sqrt{10s}}\sin\sqrt{\frac{2s}{5}} . 0\right\rangle$$

$$\mathbf{K} = \|T'(s)\| = \sqrt{\frac{16}{250s}\cos^2\sqrt{\frac{2s}{5}} + \frac{16}{250s}\sin^2\sqrt{\frac{2s}{5}}} = \sqrt{\frac{16}{250s}} = \frac{4}{5\sqrt{10s}} = \frac{4\sqrt{10s}}{50s} = \frac{2\sqrt{s}}{25s}$$

Now we are going to obtain a formula, which will help us find the curvature, when $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ **, where** t **is not the arc length parameter.**

We have

$K = \left|\left|\frac{dT}{ds}\right|\right| = \left|\left|\frac{dT}{dt} \cdot \frac{dt}{ds}\right|\right|$ **by the Chain Rule**

$$\mathbf{K} = \left|\left|\frac{\frac{dT}{dt}}{\frac{ds}{dt}}\right|\right|$$

$$\mathbf{K} = \left|\left|\frac{T'(t)}{\left|\left|\frac{d\vec{r}}{dt}\right|\right|}\right|\right|$$

$$\mathbf{K} = \frac{\|T'(t)\|}{\left|\left|\vec{r}\,'(t)\right|\right|}$$

$\rightarrow$

$$\boxed{\|T'(t)\| = K\left|\left|\vec{r}\,'(t)\right|\right|}$$

Recall that $T = \frac{\left(\frac{d\vec{r}}{dt}\right)}{\left|\left|\frac{d\vec{r}}{dt}\right|\right|}$

this gives us $\left(\frac{d\vec{r}}{dt}\right) = \left|\left|\frac{d\vec{r}}{dt}\right|\right| T$

that is $\left(\frac{d\vec{r}}{dt}\right) = \frac{ds}{dt} T$

or

$\vec{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$

$\rightarrow$

$\frac{d\vec{r}'(t)}{dt} = \vec{r}''(t) = \frac{d}{dt}\left(\frac{ds}{dt}T(t)\right)$

$\rightarrow$

$\vec{r}''(t) = \frac{d}{dt}\left(\frac{ds}{dt}T\right) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$

$\rightarrow$

$\vec{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$

$\rightarrow$

$\vec{r}'(t) \times \vec{r}''(t) = \frac{d^2s}{dt^2}\vec{r}'(t) \times \mathbf{T}(t) + \frac{ds}{dt}\vec{r}'(t) \times \mathbf{T}'(t)$

$\rightarrow$

$\vec{r}'(t) \times \vec{r}''(t) = \frac{d^2s}{dt^2}\vec{r}'(t) \times \mathbf{T}(t) + \frac{ds}{dt}\vec{r}'(t) \times \mathbf{T}'(t)$

$\rightarrow$

$\vec{r}'(t) \times \vec{r}''(t) = \frac{d^2s}{dt^2}\vec{r}'(t) \times \mathbf{T}(t) + \frac{ds}{dt}\left(\frac{ds}{dt}T\right) \times \mathbf{T}'(t)$

$\rightarrow$

$\vec{r}'(t) \times \vec{r}''(t) = \frac{d^2s}{dt^2}\left(\frac{ds}{dt}T(t)\right) \times T(t) + \frac{ds}{dt}\left(\frac{ds}{dt}T(t)\right) \times T'(t)$ **[**$\vec{r}'(t)$ **replaced by** $\frac{ds}{dt}T(t)$**]**

$\rightarrow$

$$\vec{r}'(t)\times\vec{r}''(t)=\frac{d^2s}{dt^2}\frac{ds}{dt}(T(t)\times T(t))+\frac{ds}{dt}\left(\frac{ds}{dt}T(t)\right)\times\mathbf{T}'(t)$$

$\rightarrow$

$\vec{r}'(t)\times\vec{r}''(t) = \frac{ds}{dt}\left(\frac{ds}{dt}T(t)\right)\times T'(t)$ **[the cross product of a vector with a parallel vector is 0]**

$\rightarrow$

$$\left|\left|\vec{r}'(t)\times\vec{r}''(t)\right|\right|=\left|\left|\frac{ds}{dt}\left(\frac{ds}{dt}T(t)\right)\times T'(t)\right|\right|$$

$\rightarrow$

$$\left|\left|\vec{r}'(t)\times\vec{r}''(t)\right|\right|=\left|\left|\left(\frac{ds}{dt}\right)^2T(t)\times T'(t)\right|\right|$$

$\rightarrow$

$$\left|\left|\vec{r}'(t)\times\vec{r}''(t)\right|\right| = \left(\frac{ds}{dt}\right)^2||T(t)\times T'(t)||$$

Remember that T and T' are perpendicular, therefore $||T\times T'|| = ||T||||T'||\sin\left(\frac{\pi}{2}\right) = ||T||||T'||$

That is

$$\left|\left|\vec{r}'(t)\times\vec{r}''(t)\right|\right|=\left(\frac{ds}{dt}\right)^2||T(t)||||T'(t)||$$

$\rightarrow$

$\left|\left|\vec{r}'(t)\times\vec{r}''(t)\right|\right| = \left|\left|\vec{r}'(t)\right|\right|^2||T(t)||K\left|\left|\vec{r}'(t)\right|\right|$ **because[**$\left|\left|\vec{r}'(t)\right|\right| = \frac{ds}{dt}$ **and** $T'(t) = K\left|\left|\vec{r}'(t)\right|\right|$**]**

$$\left|\left|\vec{r}'(t)\times\vec{r}''(t)\right|\right|= \mathbf{K}\left|\left|\vec{r}'(t)\right|\right|^3$$

therefore

$$\boxed{K = \frac{\left|\left|\vec{r}'(t)\times\vec{r}''(t)\right|\right|}{\left|\left|\vec{r}'(t)\right|\right|^3}}$$

Illustration:

#24 on the page 876

Find the curvature K **of the curve given by** $\vec{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \frac{3}{2}t^2 \right\rangle$

$$\vec{r}'(t) = \langle 4(\cos t + t\sin t - \cos t), 4(-\sin t + t\cos t + \sin t), 3t\rangle$$

that is

$$\vec{r}'(t) = \langle 4t\sin t, 4t\cos t, 3t\rangle$$

$$\vec{r}''(t) = \langle 4(t\cos t + \sin t), 4(-t\sin t + \cos t), 3\rangle$$

$$\vec{r}'(t) \times \vec{r}''(t)$$

$$= \begin{vmatrix} i & j & k \\ 4t\sin t & 4t\cos t & 3t \\ 4(t\cos t + \sin t) & 4(-t\sin t + \cos t) & 3 \end{vmatrix}$$

$$= ((4t\cos t)3 - (4(-t\sin t + \cos t)3t))\mathbf{i} + (3t(4(t\cos t + \sin t)) - 4t\sin t(3))\mathbf{j} + ((4t\sin t)(4(-t\sin t + \cos t)) - (4t\cos t)(4(t\cos t + \sin t)))$$

$$= (12t\cos t + 12t^2\sin t - 12t\cos t)\mathbf{i} + (12t^2\cos t + 12t\sin t - 12t\sin t)\mathbf{j} + \left(-16t^2\sin^2 t + 16t\sin t\cos t - 16t^2\cos^2 t - 16\cos t\sin t\right)\mathbf{k}$$

$$= (12t^2\sin t)\mathbf{i} + (12t^2\cos t)\mathbf{j} + \left(-16t^2\sin^2 t - 16t^2\cos^2 t\right)\mathbf{k}$$

$$= (12t^2\sin t)\mathbf{i} + (12t^2\cos t)\mathbf{j} - \mathbf{16t}^2\left(\sin^2 t + \cos^2 t\right)\mathbf{k}$$

$$= (12t^2\sin t)\mathbf{i} + (12t^2\cos t)\mathbf{j} - \mathbf{16t}^2\mathbf{k}$$

$$\left|\left|\vec{r}'(t) \times \vec{r}''(t)\right|\right| = \sqrt{(12t^2\sin t)^2 + (12t^2\cos t)^2 + (-16t^2)^2}$$

$$\left|\left|\vec{r}'(t) \times \vec{r}''(t)\right|\right| = \sqrt{144t^4\left(\sin^2 t + \cos^2 t\right) + 256t^4}$$

$$\left|\left|\vec{r}'(t) \times \vec{r}''(t)\right|\right| = \sqrt{144t^4 + 256t^4}$$

$$\boxed{\left|\left|\vec{r}'(t) \times \vec{r}''(t)\right|\right| = \sqrt{400t^2} = 20t^2}$$

$$\left|\left|\vec{r}'(t)\right|\right| = \sqrt{16t^2\left(\sin^2 t + \cos^2 t\right) + 9t^2}$$

$$\left|\left|\vec{r}'(t)\right|\right| = \sqrt{16t^2 + 9t^2}$$

$$\left|\left|\vec{r}'(t)\right|\right| = \sqrt{25t^2}$$

$\left|\left|\vec{r}'(t)\right|\right| =$ **5t**

Therefore

K $= \frac{20t^2}{125t^3} = \frac{4}{25t}$

$\frac{4}{25t} = \frac{4}{25}\sqrt{\frac{5}{2s}} = \frac{2\sqrt{10s}}{25s}$ **, the two results are the same.**

....

Consider the motion along a smooth curve traced by $\vec{r}(t)$

The velocity vector is

$$\vec{v}=\frac{d\vec{r}}{dt}=\left|\left|\frac{d\vec{r}}{dt}\right|\right|\frac{\left(\frac{d\vec{r}}{dt}\right)}{\left|\left|\frac{d\vec{r}}{dt}\right|\right|}$$

That is

$\vec{v} = \frac{ds}{dt}T$ **because** $\left|\left|\frac{d\vec{r}}{dt}\right|\right| = \frac{ds}{dt}$ **and** $\frac{\left(\frac{d\vec{r}}{dt}\right)}{\left|\left|\frac{d\vec{r}}{dt}\right|\right|} = T$

the acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2s}{dt^2}T + \left(\frac{ds}{dt}\right)\frac{dT}{dt}$

Because the normal vector $N = \frac{\frac{dT}{dt}}{\left|\left|\frac{dT}{dt}\right|\right|}$

$$\vec{a}=\frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)\left|\left|\frac{dT}{dt}\right|\right|\mathbf{N}$$

we saw above that $\|T'(t)\| = K\left|\left|\vec{r}\,'(t)\right|\right| = K\frac{ds}{dt}$

$$\vec{a}=\frac{d^2s}{dt^2}\mathbf{T} + \mathbf{K}\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

We have

Tangential Component of Acceleration$=\frac{d^2s}{dt^2}$ **and normal component of acceleration**$=K\left(\frac{ds}{dt}\right)^2$

Read the pages 867-875 and work on the following practice problems:

Section 12.5:

9,13,19,23,39,45,85