**Lesson 4 Part 1**

**This lesson corresponds to the section 12.4 in the text book.**

**Remember that for a smooth curve** $\vec{r}(t)$**, if** $\vec{r}\,'(t) \neq 0$

$\frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$ **is the unit tangent vector with the notation** $T(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$

**Take the example of** $\vec{r}(t) = 3\cos(t)i + 2\sin(t)j$

$\vec{r}\,'(t) = -\mathbf{3}\sin(t)\mathbf{i} + \mathbf{2}\cos(t)\mathbf{j}$

$||\vec{r}\,'(t)|| = \sqrt{9\sin^2 t + 4\cos^2 t} \neq \mathbf{0}$

**To find the unit tangent vector at the point corresponding to** $t = \frac{\pi}{6}$

**at** $t = \frac{\pi}{6}$

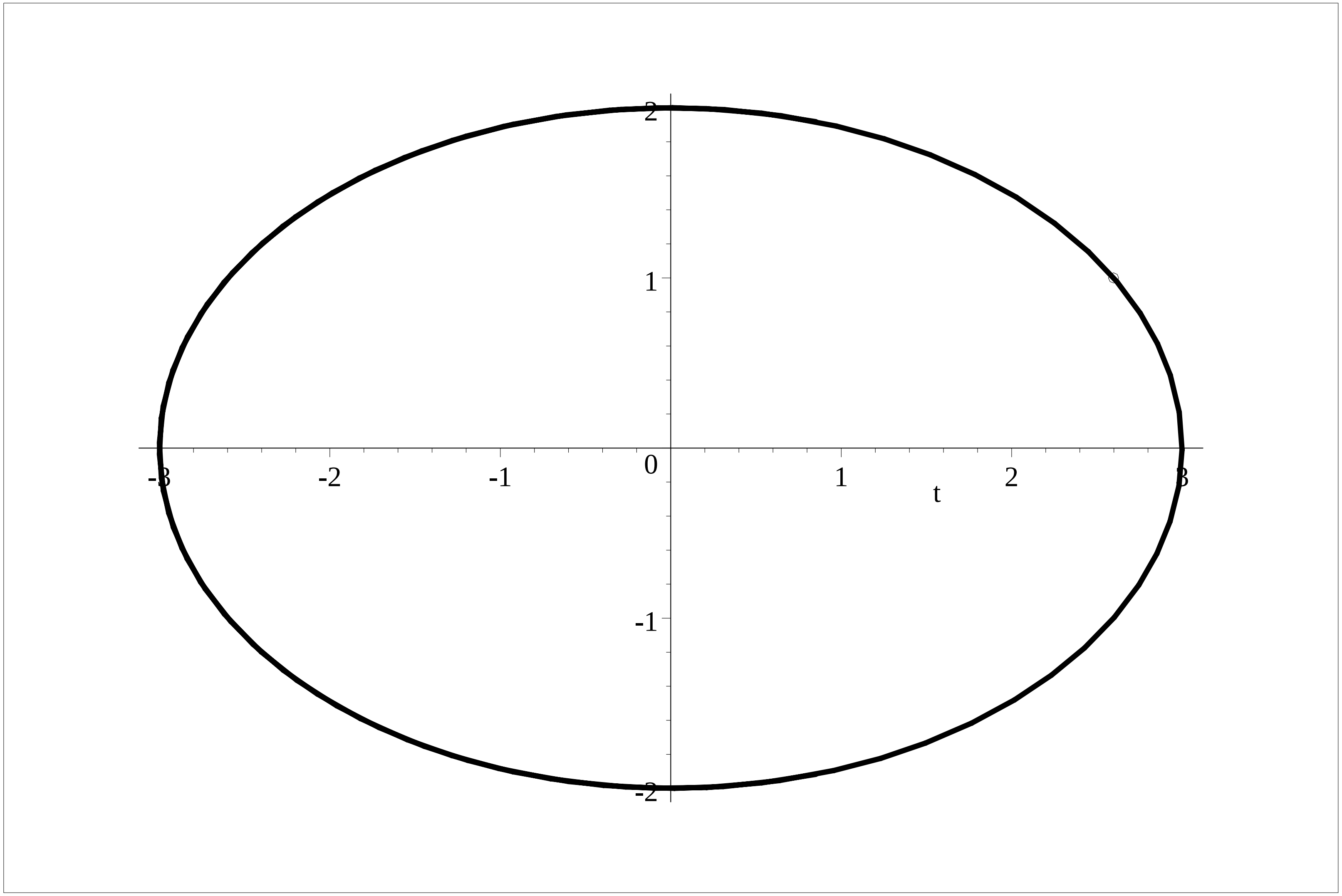
$\vec{r}\,'\left(\frac{\pi}{6}\right) = -3\sin\left(\frac{\pi}{6}\right)i + 2\cos\left(\frac{\pi}{6}\right)j$

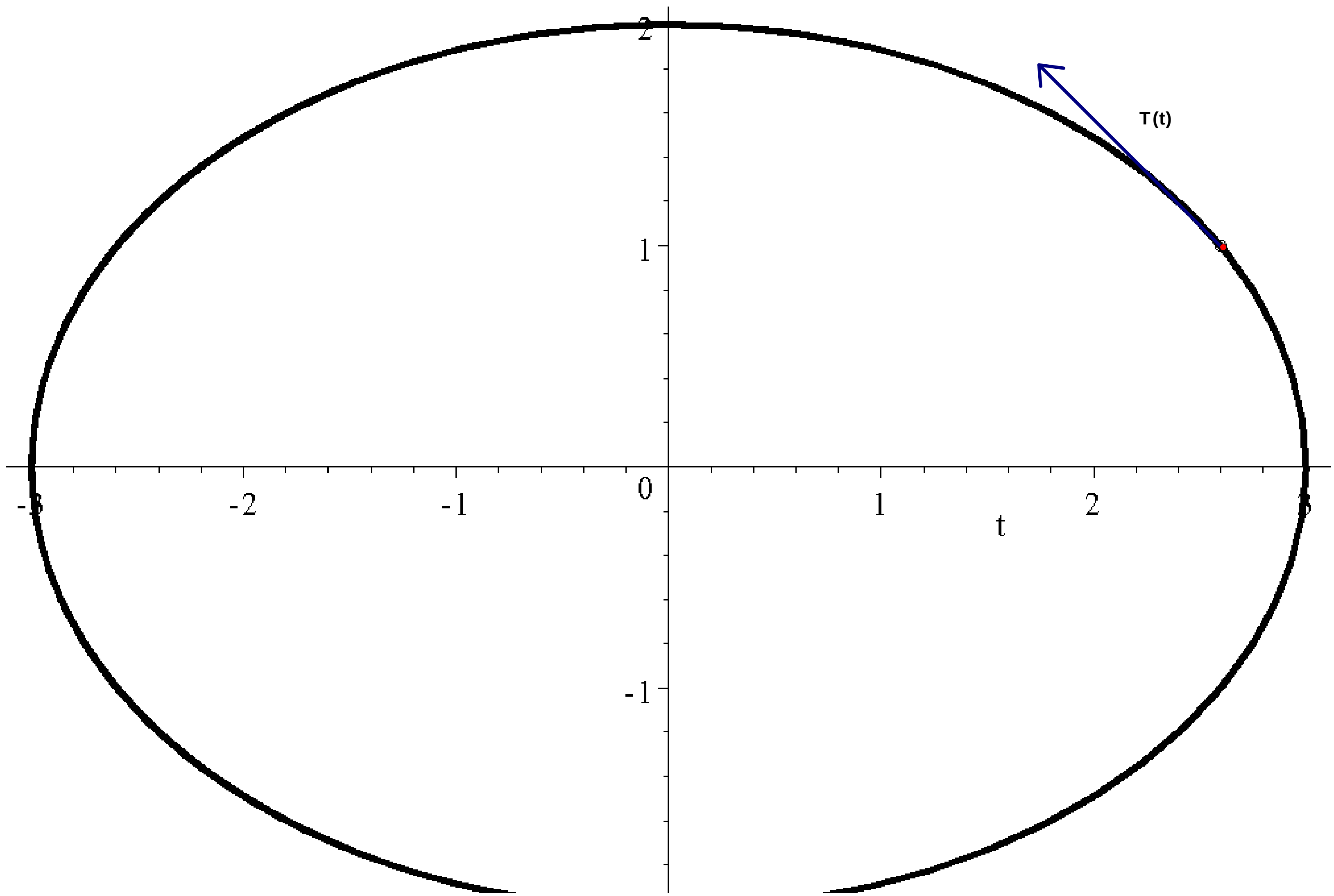
$\vec{r}\,'\left(\frac{\pi}{6}\right) = -\frac{3}{2}\mathbf{i} + \mathbf{2}\frac{\sqrt{3}}{2}\mathbf{j}$

$\vec{r}\,'\left(\frac{\pi}{6}\right) = -\frac{3}{2}\mathbf{i} + \sqrt{3}\,\mathbf{j}$

$||\vec{r}\,'(t)|| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\sqrt{3}\right)^2} = \frac{1}{2}\sqrt{21}$

$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{-\frac{3}{2}i + \sqrt{3}j}{\frac{1}{2}\sqrt{21}} = \frac{-3i + 2\sqrt{3}j}{\sqrt{21}}$


2
1
0
-3
-2
-1
1
2
3
t
-1
-2



T(t)
2
1
-1
0
-3
-2
-1
1
2
3
t

**Example 2:**

**To find a parametric equation of tangent line to a graph of**

$\vec{r}(t) = \langle t\cos t, t + \sin t, 2t\rangle$ **at** $P(-\pi, \pi, 2\pi)$

**Note that** $2t = 2\pi \rightarrow t = \pi$
$\boldsymbol{\pi}\cos\boldsymbol{\pi} = -\boldsymbol{\pi}$
$\boldsymbol{\pi} + \sin\boldsymbol{\pi} = \boldsymbol{\pi}$

**therefore the point corresponds to** $t = \pi$

$\vec{r}'(t) = \langle\cos t - t\sin t, 1 + \cos t, 2\rangle$

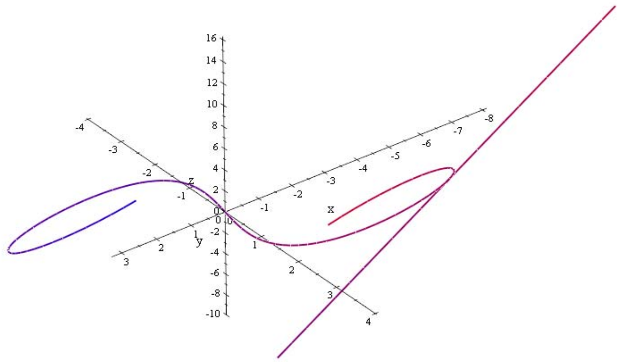
$\vec{r}'(\pi) = \langle\cos\pi - \pi\sin\pi, 1 + \cos\pi, 2\rangle$
$\vec{r}'(\pi) = \langle -1, 0, 2\rangle$ **a vecor along the tangent line at** $P(-\pi, \pi, 2\pi)$

**Therefore the tangent line passes through** $P(-\pi, \pi, 2\pi)$ **and is parallel to** $\langle -1, 0, 2\rangle$

**therefore a parametric equation of the tangent line at** $P(-\pi, \pi, 2\pi)$ **to the space curve** $\vec{r}(t) = \langle t\cos t, t + \sin t, 2t\rangle$ **is**

$\mathbf{x} = -\boldsymbol{\pi} - \mathbf{t}$ $\quad$ $\mathbf{y} = \boldsymbol{\pi}$ $\quad$ $\mathbf{z} = \mathbf{2}\boldsymbol{\pi} + \mathbf{2t}$

$[t\cos t, t + \sin t, 2t]$



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Since $T(t)$ is a unit vector, note that

$\mathbf{T}(t)\cdot\mathbf{T}(t) = 1$
differentiate
$\mathbf{T}(t)\cdot\mathbf{T}'(t)+\mathbf{T}'(t)\cdot\mathbf{T}(t) = 0$

$2\mathbf{T}(t)\cdot\mathbf{T}'(t) = 0$

Therefore $T(t)$ and $T'(t)$ are orthogonal

**for a smooth curve** $\vec{r}(t)$

$N(t) = \frac{T'(t)}{\|T'(t)\|}$ **is called the principal unit normal vector at** $t$

**provided that** $T'(t) \neq 0$

**Example 3:**

**To find the principal unit normal vector to the curve** $\vec{r}(t) = e^{-t}i + e^{t}j + \sqrt{2}\,tk$

**at** $t = 0$

$\vec{r}\,'(t) = -e^{-t}i + e^{t}j + \sqrt{2}\,k$

$\mathbf{T}(t)=\frac{\vec{r}\,'(t)}{\left|\left|\vec{r}\,'(t)\right|\right|}$

$\left|\left|\vec{r}\,'(t)\right|\right|=\sqrt{e^{-2t} + e^{2t} + 2}$

$\left|\left|\vec{r}\,'(t)\right|\right|=\sqrt{(e^{-t} + e^{t})^2}$

$\left|\left|\vec{r}\,'(t)\right|\right|= \mathbf{e}^{-t}+\mathbf{e}^{t}$

$\mathbf{T}(t)=\frac{\vec{r}\,'(t)}{\left|\left|\vec{r}\,'(t)\right|\right|}$

$\rightarrow$

$\mathbf{T}(t)=\frac{-e^{-t}i+e^{t}j+\sqrt{2}\,k}{e^{-t}+e^{t}}=\frac{e^{t}\left(-e^{-t}i+e^{t}j+\sqrt{2}\,k\right)}{e^{t}(e^{-t}+e^{t})}=\frac{-i+e^{2t}j+\sqrt{2}\,e^{t}k}{1+e^{2t}}$

$\mathbf{T}(t)= -\frac{1}{1+e^{2t}}\mathbf{i}+\frac{e^{2t}}{1+e^{2t}}\mathbf{j}+\frac{\sqrt{2}\,e^{t}}{1+e^{2t}}\mathbf{k}$

$T'(t) = \frac{2e^{2t}}{(1+e^{2t})^2}i+\frac{2e^{2t}(1+e^{2t})-2e^{2t}(2e^{2t})}{1+e^{2t}}j+\sqrt{2}\,\frac{e^{t}(1+e^{2t})-2e^{2t}e^{t}}{1+e^{2t}}k$ **(Quotient and Reciprocal Rules)**

$\mathbf{T}'(t)=\frac{2e^{2t}}{(1+e^{2t})^2}\mathbf{i}+\frac{2e^{2t}-4e^{4t}}{1+e^{2t}}\mathbf{j}+\sqrt{2}\,\frac{e^{t}-e^{3t}}{1+e^{2t}}\mathbf{k}$

$T'(0) = \frac{1}{2}i+\frac{1}{2}j$

$||T'(0)||=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{\sqrt{2}}$

$\mathbf{N}(0)=\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j}$

**Let us write equations of the lines that are along the tangent and the principal normal to the graph of**

$\vec{r}(t) = e^{-t}i+e^{t}j+\sqrt{2}\,tk$ **at the point that corresponds to** $t = 0$

**The point that corresponds to** $t = 0$ **has coordinates**

$\left(e^{-0}, e^{0}, \sqrt{2}\times 0\right)=(1,1,0)$

**The tangent line is parralel to**

$\mathbf{T}(0)= -\frac{1}{1+e^{2(0)}}\mathbf{i}+\frac{e^{t(0)}}{1+e^{2(0)}}\mathbf{j}+\frac{\sqrt{2}\,e^{(0)}}{1+e^{t(0)}}\mathbf{k}$

**That is**

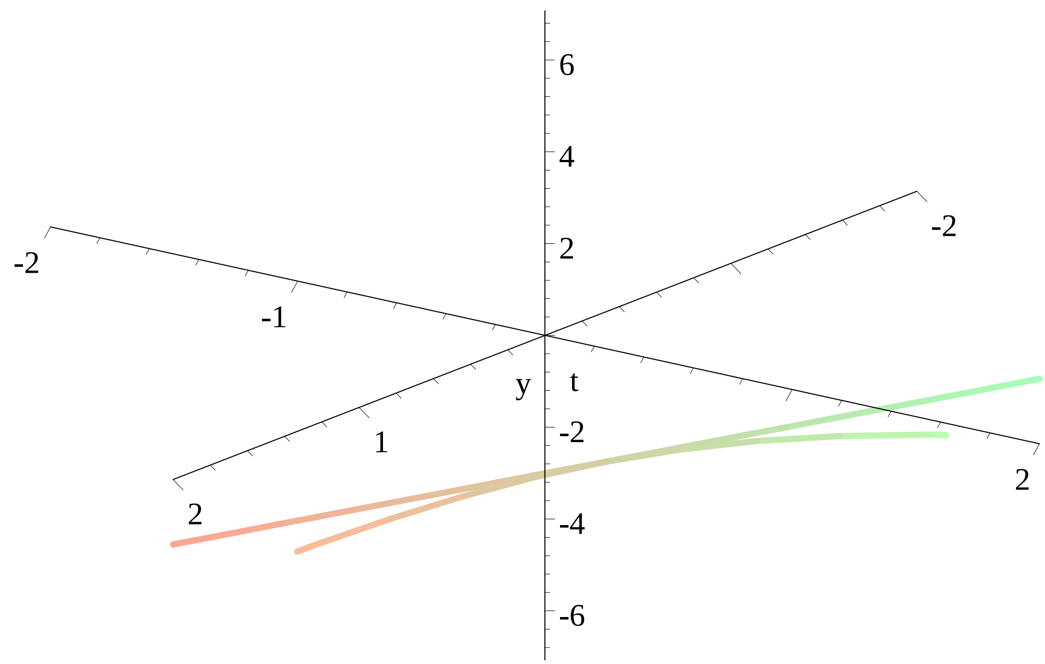
**T**$(0) = -\frac{1}{2}$**i** $+\frac{1}{1+1}$**j** $+\frac{\sqrt{2}}{1+1}$**k**

**or**

$\left\langle \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right\rangle$

**Parametric Equation of a line that passes through** $(1, 1, 0)$ **and is parallel to** $\left\langle \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right\rangle$

**is** $x = 1 - \frac{1}{2}t$ **,** $y = 1 + \frac{1}{2}t$**,** $z = \frac{\sqrt{2}}{2}t$


6
4
2
-2
-4
-6
-2
-1
1
2
-2
2
y
t

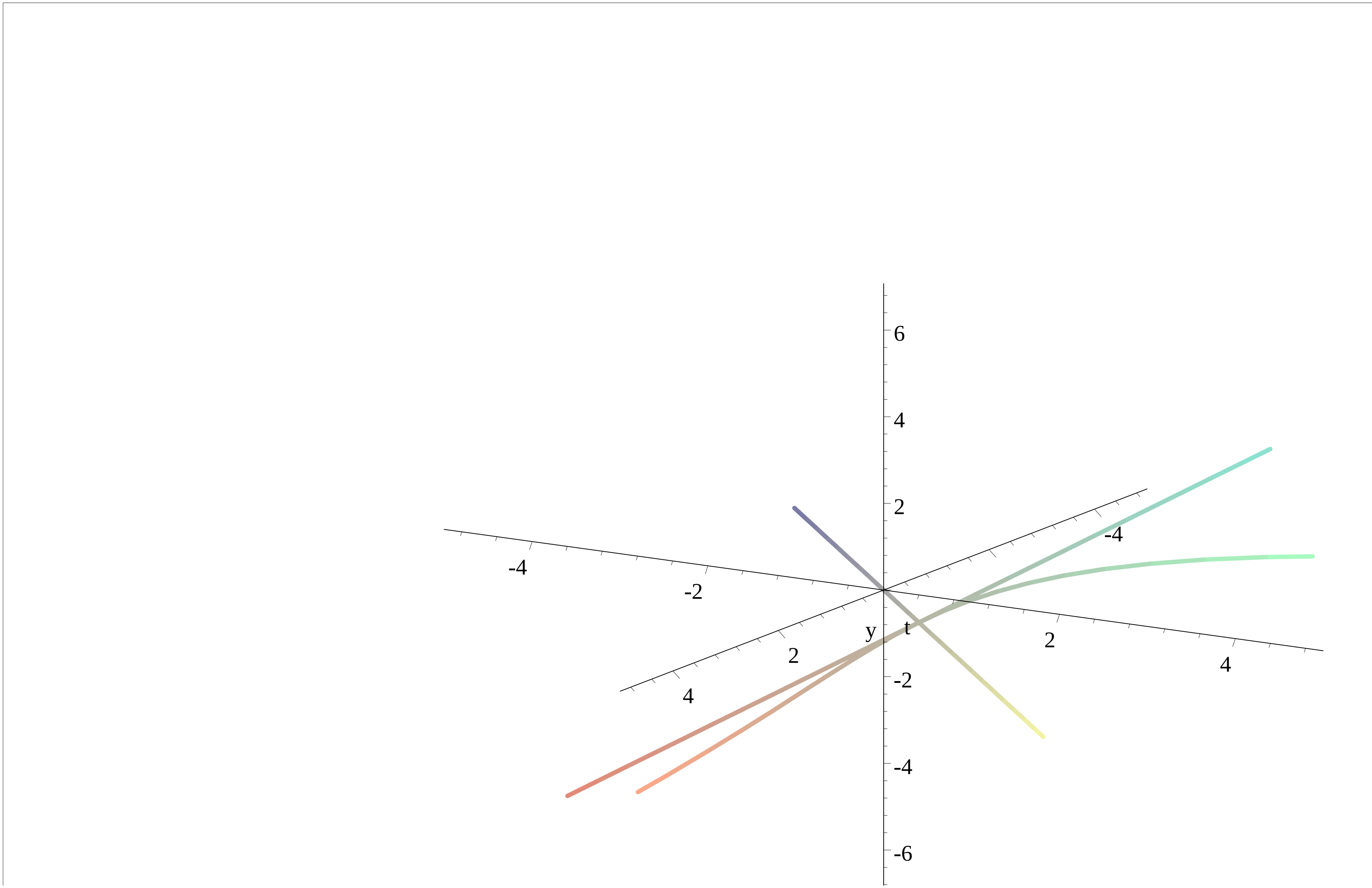
**Recall from above that the normal at** $(1,1,0)$ **is parallel to** $N(0) = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j$

**Parametric equation of a line that passes through** $(1,1,0)$ **and is parallel to**

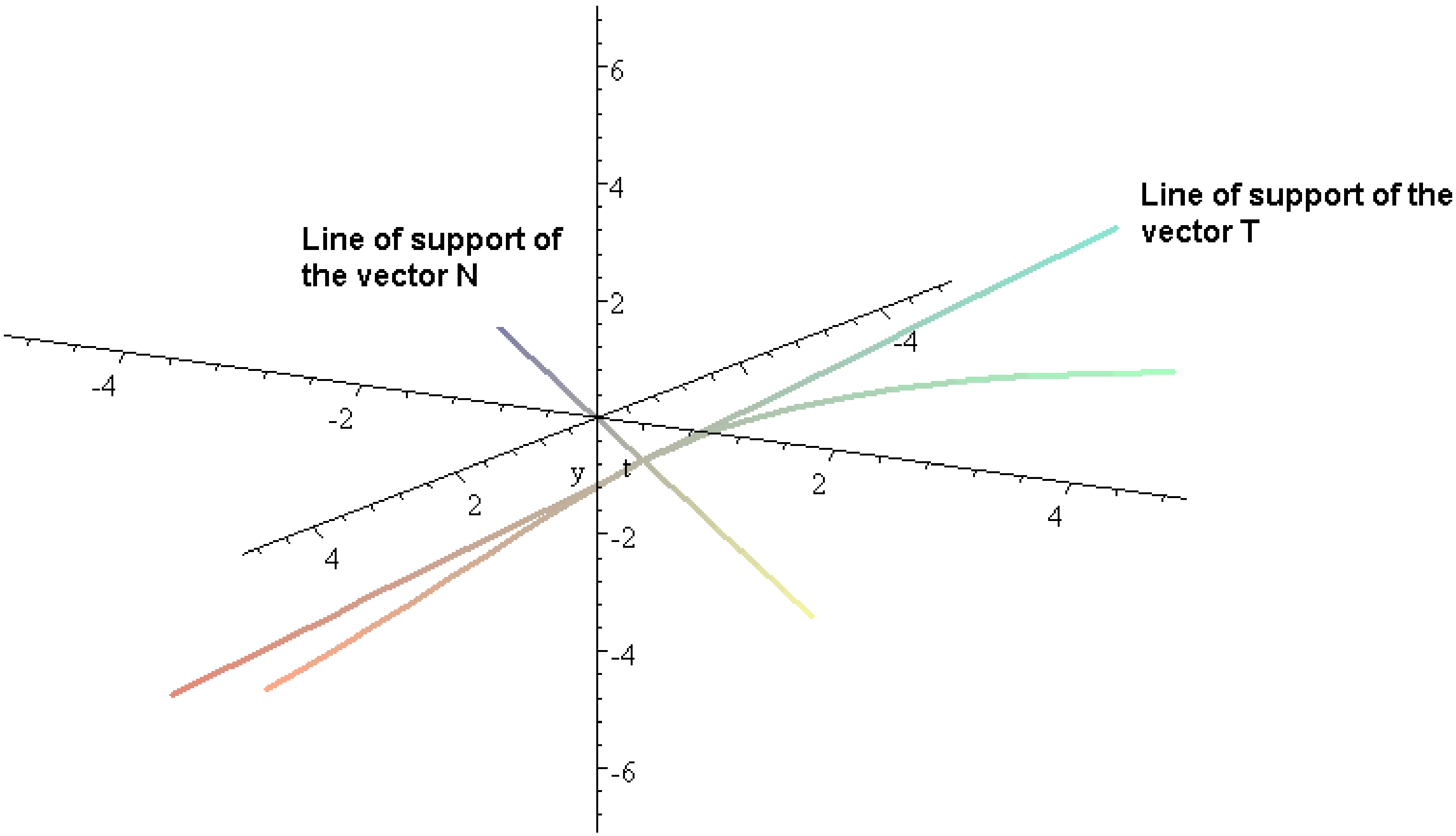
$\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j$ **is**

$x = 1 + \frac{\sqrt{2}}{2}t$, $y = 1 + \frac{\sqrt{2}}{2}t$, $z = 0$

**our calculations are confirmed by the graph,**



6
4
2
-2
-4
-6
-4
-2
2
4
y
t
2
4
-4




**Discussion of the acceleration vector:**

**Remeber that the velocity vector is**

$$\vec{v}=\frac{d\vec{r}}{dt}$$

**and the unit tangent vector is** $T = \frac{\frac{d\vec{r}}{dt}}{\left|\left|\frac{d\vec{r}}{dt}\right|\right|}$ **if** $\left|\left|\frac{d\vec{r}}{dt}\right|\right| \neq 0$

$$\vec{v} = \frac{d\vec{r}}{dt}$$
$$\Longrightarrow$$
$$\vec{v}=\left|\left|\frac{d\vec{r}}{dt}\right|\right|\frac{\frac{d\vec{r}}{dt}}{\left|\left|\frac{d\vec{r}}{dt}\right|\right|}$$
$$\Longrightarrow$$
$$\vec{v}=\left|\left|\vec{v}\right|\right|\mathbf{T}$$
$$\Longrightarrow$$
$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d\left|\left|\vec{v}\right|\right|}{dt}\mathbf{T}+\left|\left|\vec{v}\right|\right|\frac{dT}{dt}$$

**Recall that** $N = \frac{\frac{dT}{dt}}{\left|\left|\frac{dT}{dt}\right|\right|}$

$$\vec{a}=\frac{d\left|\left|\vec{v}\right|\right|}{dt}\mathbf{T}+\left|\left|\vec{v}\right|\right|\left|\left|\frac{dT}{dt}\right|\right|\frac{\frac{dT}{dt}}{\left|\left|\frac{dT}{dt}\right|\right|}$$

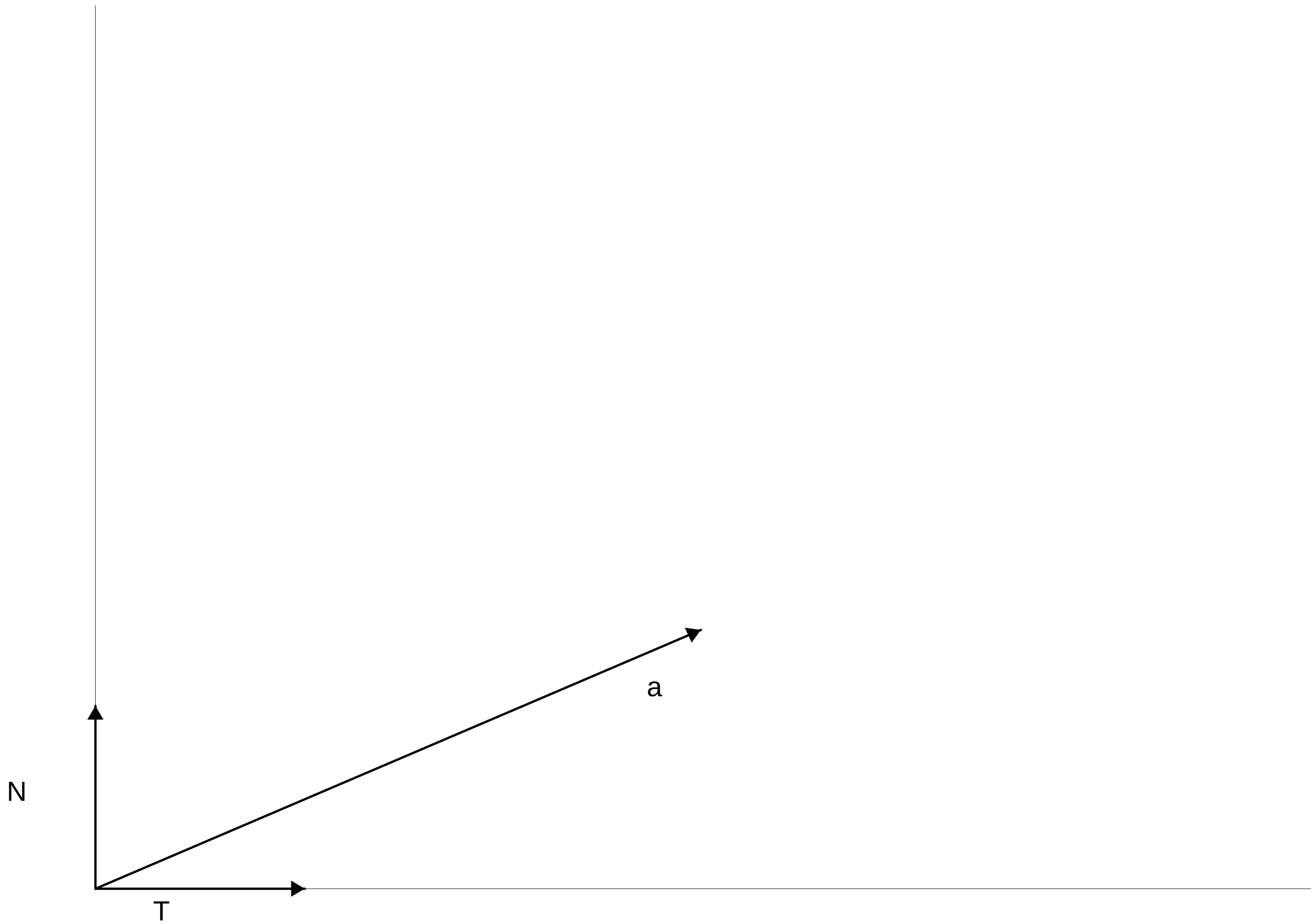
**or**

$$\vec{a}=\frac{d\left|\left|\vec{v}\right|\right|}{dt}\mathbf{T}+\left|\left|\vec{v}\right|\right|\left|\left|\frac{dT}{dt}\right|\right|\mathbf{N}$$

**Therefore the acceleration vector can always be written in terms of the vectors** $T$ **and** $N$

**therefore $\vec{a}$ lies in the plane of $T$ and $N$**

**Let us find the components of acceleration along the tangent and the normal**




**Component of** $\vec{a}$ **along** $T$ **is given by** $\vec{a} \cdot T$ **,it is denoted by** $a_T$

**Component along** $N$ **is** $\vec{a} \cdot N$ **or the same as** $\left|\left|\vec{a} - \left(\vec{a} \cdot T\right)T\right|\right|$**, ,it is denoted by** $a_N$

**We may prove that** $\boxed{a_T = \frac{\vec{v}\cdot\vec{a}}{||\vec{v}||} \text{ and } a_N = \frac{|\vec{v}\times\vec{a}|}{||\vec{v}||}}$

**As an illustration, let us take up #56 on the page 864 of the text book, where**

**we are given**

$\vec{r}(t) = \langle e^t \sin t, e^t \cos t, e^t \rangle$

**we have to find** $T(t), N(t), \vec{a}_T, \vec{a}_N$ **that corresponds to** $t = 0$

$\vec{v} = \frac{d\vec{r}}{dt} = \langle e^t \cos t + e^t \sin t, -e^t \sin t + e^t \cos t, e^t \rangle$

$$\left|\left|\frac{d\vec{r}}{dt}\right|\right|$$
$$= \sqrt{(e^t \cos t + e^t \sin t)^2 + (-e^t \sin t + e^t \cos t)^2 + (e^t)^2}$$
$$= \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2 t + 2e^t e^t \cos t \sin t + e^{2t}\sin^2 t + e^{2t}\cos^2 t - 2e^t e^t \cos t \sin t + e^{2t}}$$
$$= \sqrt{e^{2t}\left(\cos^2 t + \sin^2 t\right) + e^{2t}\left(\sin^2 t + \cos^2 t\right) + e^{2t}}$$
$$= \sqrt{e^{2t} + e^{2t} + e^{2t}}$$
$$= \sqrt{3e^{2t}}$$
$$= \sqrt{3}\, e^t$$

$$\mathbf{T}(t)$$
$$= \frac{1}{\sqrt{3}\, e^t}\langle e^t \cos t + e^t \sin t, -e^t \sin t + e^t \cos t, e^t \rangle$$
$$= \frac{1}{\sqrt{3}}\langle \cos t + \sin t, -\sin t + \cos t, 1 \rangle$$

$$\frac{dT}{dt} = \frac{1}{\sqrt{3}}\langle -\sin t + \cos t, -\cos t - \sin t, 0 \rangle$$

$$\left|\left|\frac{dT}{dt}\right|\right|$$
$$=\frac{1}{\sqrt{3}}\sqrt{(-\sin t+\cos t)^2+(-\cos t-\sin t)^2+0^2}$$
$$=\frac{1}{\sqrt{3}}\sqrt{\sin^2 t+\cos^2 t-2\sin t\cos t+\cos^2 t+\sin^2 t+2\sin t\cos t}$$
$$=\frac{1}{\sqrt{3}}\sqrt{2}$$

$$\mathbf{N}(t)=\frac{\frac{1}{\sqrt{3}}\langle -\sin t+\cos t,-\cos t-\sin t,0\rangle}{\frac{1}{\sqrt{3}}\sqrt{2}}=\frac{1}{\sqrt{2}}\langle -\sin t+\cos t,-\cos t-\sin t,0\rangle$$

$$\vec{v}=\frac{d\vec{r}}{dt}=\langle e^t\cos t+e^t\sin t,-e^t\sin t+e^t\cos t,e^t\rangle$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\langle e^t\cos t-e^t\sin t+e^t\cos t+e^t\sin t,-e^t\cos t-e^t\sin t+e^t\cos t-e^t\sin t,e^t\rangle$$

$$\vec{a}=\langle 2e^t\cos t,-2e^t\sin t,e^t\rangle$$

$$\mathbf{a}_T=\vec{a}\cdot\mathbf{T}(t)$$

$$\mathbf{a}_T$$
$$=\langle 2e^t\cos t,-2e^t\sin t,e^t\rangle\cdot\frac{1}{\sqrt{3}}\langle\cos t+\sin t,-\sin t+\cos t,1\rangle$$
$$=\frac{1}{\sqrt{3}}[2e^t\cos t(\cos t+\sin t)-2e^t\sin t(-\sin t+\cos t)+e^t]$$
$$=\frac{1}{\sqrt{3}}\left[2e^t\cos^2 t+2e^t\cos t\sin t+2e^t\sin^2 t-2e^t\sin t\cos t+e^t\right]$$
$$=\frac{1}{\sqrt{3}}\left[2e^t\cos^2 t+2e^t\sin^2 t+e^t\right]$$
$$=\frac{1}{\sqrt{3}}\left[2e^t\left(\cos^2 t+\sin^2 t\right)+e^t\right]$$
$$=\frac{1}{\sqrt{3}}[2e^t+e^t]$$
$$=\frac{1}{\sqrt{3}}(3e^t)$$

$=\sqrt{3}\, e^t$

$\mathbf{a}_N$

$=\vec{a} \cdot \mathbf{N}(t)$

$=\langle 2e^t \cos t, -2e^t \sin t, e^t\rangle \cdot \frac{1}{\sqrt{2}}\langle -\sin t + \cos t, -\cos t - \sin t, 0\rangle$

$=\frac{1}{\sqrt{2}}[2e^t \cos t(-\sin t + \cos t) - 2e^t \sin t(-\cos t - \sin t) + 0]$

$=\frac{1}{\sqrt{2}}\left[-2e^t \cos t \sin t + 2e^t \cos^2 t + 2e^t \cos t \sin t + 2e^t \sin^2 t + 0\right]$

$=\frac{1}{\sqrt{2}}\left[2e^t \cos^2 t + 2e^t \sin^2 t\right]$

$=\frac{1}{\sqrt{2}}\left(2e^t\left(\cos^2 t + \sin^2 t\right)\right)$

$=\frac{1}{\sqrt{2}} 2e^t$

$=\sqrt{2}\, e^t$

**We have found**

$\mathbf{T}(t)=\frac{1}{\sqrt{3}}\langle \cos t + \sin t, -\sin t + \cos t, 1\rangle$

$\mathbf{N}(t)=\frac{1}{\sqrt{2}}\langle -\sin t + \cos t, -\cos t - \sin t, 0\rangle$

$\mathbf{a}_T=\sqrt{3}\, \mathbf{e}^t$

$\mathbf{a}_N=\sqrt{2}\, \mathbf{e}^t$

**at** $t = 0$

$\mathbf{T} =\frac{1}{\sqrt{3}}\langle \cos 0 + \sin 0, -\sin 0 + \cos 0, 1\rangle=\frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle$

$\mathbf{N} =\frac{1}{\sqrt{2}}\langle -\sin 0 + \cos 0, -\cos 0 - \sin 0, 0\rangle=\frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle$

$\mathbf{a}_T=\sqrt{3}$

$\mathbf{a}_N=\sqrt{2}$

$\boxed{\text{YOU COULD USE THE HINT THAT THE BOOK PROVIDED FOR THIS PROBLEM}}$

**Suggested Practice Problems in the section 12.4**

**15,17,19,21,25,27,33,45,47,55,63,69,71**