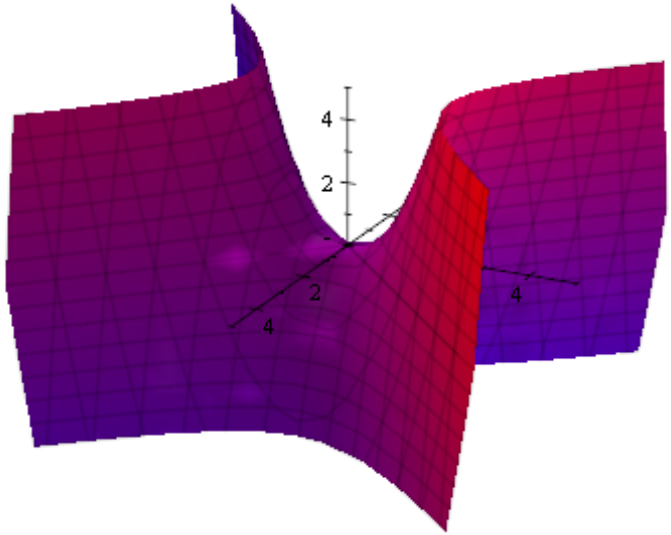
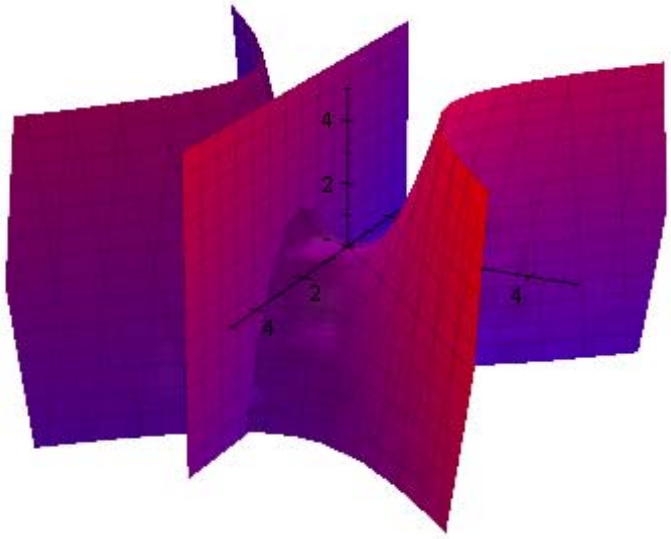
Partial Derivatives:

Consider the function: $f(x,y) = y^2 - x^2$



We would like to see the rate of change of this function, when we keep one of the two independent variables as a constant.

For example, in the above function, if we set y as a constant, the graph will be the curve formed by the intersection of the two surfaces shown below



To find the rate of change of this function along this particlar curve, we shall treat the y-value as that particular constant and look at the rate of change in wrt x

that may be performed by differentiating $f(x,y)$ **wrt** x **treating** y **as a constant.**

in case, the following limit is available:

Lim$_{\Delta x \to 0} \frac{f(x+\Delta x,y)-f(x,y)}{x}$

the notation for this limit is $\frac{\partial f}{\partial x}$

Example 1: Find $\frac{\partial f}{\partial x}$ **and** $\frac{\partial f}{\partial y}$ **if** $f(x,y) = x\sin y^2$

When we compute $\frac{\partial f}{\partial x}$, we treat y as a constant therefore $\sin y^2$ as a constant and $\frac{\partial f}{\partial x} = \sin y^2$

For $\frac{\partial f}{\partial y}$, we treat x as a constant, therefore

$\frac{\partial f}{\partial y} = x\,\frac{\partial}{\partial y}(\sin y^2) = x(2y\cos y^2)$ **, Note that we used the chain rule and also that $\frac{\partial}{\partial y}(\sin y^2)$ has the same value as $\frac{d}{dy}(\sin y^2)$ does.**

Example 2:

Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ if it is given that $f(x,y,z) = x^{y^3 z}$

For the computation of $\frac{\partial f}{\partial x}$, note that y and z will be treated as constants therefore $y^3 z$ is a constant and we are differentiating x to a constant power.

Therefore the calculation is fairly simple,

$\frac{\partial f}{\partial x} = (y^3 z)x^{y^3 z - 1}$ Because, $\frac{dx^n}{dx} = nx^{n-1}$, in this case the constant $n = y^3 z$

For the computation of $\frac{\partial f}{\partial y}$, note that x and z will be treated as constants

Now, we have a different procedure, in the case of $\frac{\partial f}{\partial x}$ for $f(x,y,z) = x^{y^3 z}$, we had a variable base and a constant power. For $\frac{\partial f}{\partial y}$ with $f(x,y,z) = x^{y^3 z}$ we have a constant base and a variable power.

Remember that $\boxed{\frac{da^u}{dt} = a^u(\ln a)\frac{du}{dt} \text{ for a constant } a > 0}$

$\frac{\partial}{\partial y}\left(x^{y^3z}\right) = \mathbf{x}^{y^3z}(\ln x)\frac{\partial}{\partial y}(y^3z) = \mathbf{x}^{y^3z}(\ln x)(3y^2z) = \mathbf{3y^2z}\left(x^{y^3z}\right)\ln \mathbf{x}$

.....

Higher order partial derivatives follow the very much the same style that the derivatives of the functions of a single variable do. But, we shall keep noticing the additional requirements as well as assumptions.

Example 3:

To find $\frac{\partial^2 f}{\partial x^2}$

for $f(x,y,z) = \sin(x^3ye^z)$

$\frac{\partial f}{\partial x} = 3x^2ye^z\cos(x^3ye^z)$ **used the Chain Rule for differentiation wrt** x **, treated** ye^z **as a constant**

For

$\frac{\partial f}{\partial x}$

we shall differentiate

$\frac{\partial f}{\partial x} = 3x^2ye^z\cos(x^3ye^z)$ **with respect to** x

we shall use the product rule combined with the chain rule and ye^z **is treated as a contstant**

$\frac{\partial^2 f}{\partial x^2}$

$= \mathbf{3}(2x)\mathbf{ye}^{z}\cos(x^3ye^z)+\mathbf{3x}^2\mathbf{ye}^{z}(-3x^2ye^z\sin(x^3ye^z))$

$= \mathbf{6xye}^{z}\cos(x^3ye^z)-\mathbf{9x}^4\mathbf{y}^2\mathbf{e}^{2z}\sin(x^3ye^z)$

Example 4:

To evaluate $\frac{\partial^2 f}{\partial x \partial y}$ **for** $f(x,y) = xe^y$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(xe^y) = \mathbf{e}^y$

A partial derivative like $\frac{\partial^2 f}{\partial x \partial y}$ **is called a MIXED PARTIAL DERIVATIVE**

In case both $\frac{\partial^2 f}{\partial x \partial y}$ **and** $\frac{\partial^2 f}{\partial y \partial x}$ **are continous on an open disk D,**

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ **at each point in D.**

Example 5:

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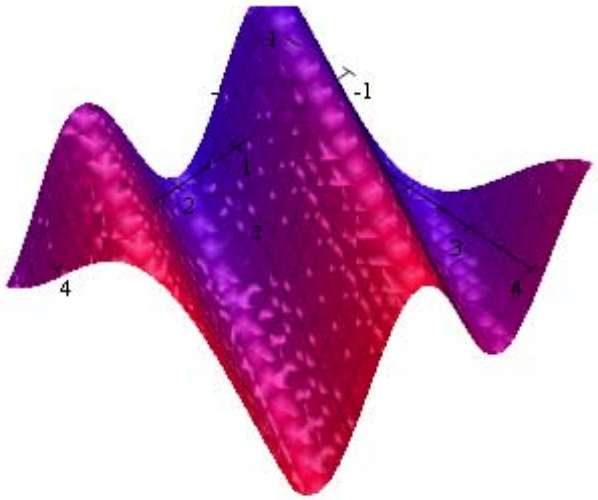
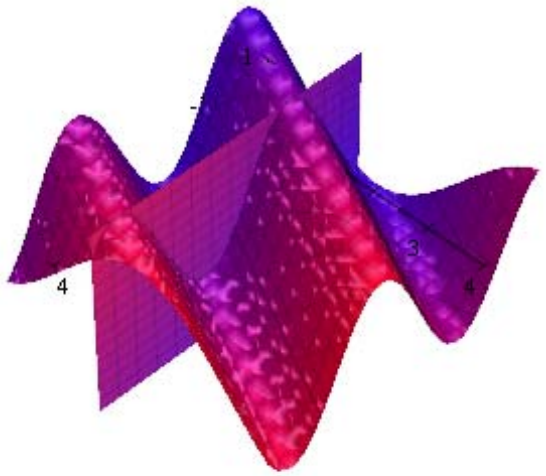
For the function $z = \cos(2x - y)$



To find the slopes of the surface in the x-direction and in the y-direction at the point $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

The slope in the x-driection at $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ **is the slope of the tangent line to the curve that is formed by taking the intersection of the surface** $z = \cos(2x - y)$ **with the plane** $y = \frac{\pi}{3}$



The slope is obtained by evaluating $\frac{\partial z}{\partial x}$ **at** $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

For $z = \cos(2x - y)$

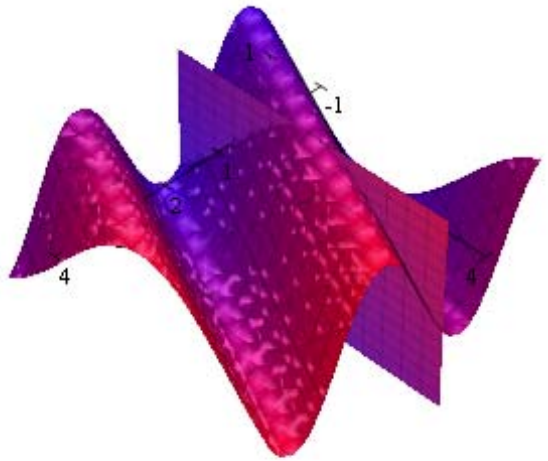
$\frac{\partial z}{\partial x} = -2\sin(2x - y)$

at $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

$\frac{\partial z}{\partial x}\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)}$ **is a notation for the value of** $\frac{\partial z}{\partial x}$ **at** $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

$\frac{\partial z}{\partial x}\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)} = -2\sin\left(2\left(\frac{\pi}{4}\right) - \frac{\pi}{3}\right) = -1$

The slope in the y-driection at $\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ **is the slope of the tangent line to the curve that is formed by taking the intersection of the surface** $z = \cos(2x - y)$ **with the plane** $x = \frac{\pi}{4}$



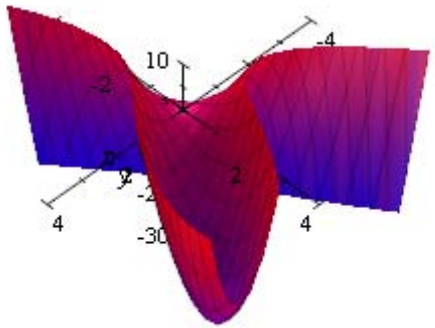
$\mathbf{z} = \cos(2x - y) \rightarrow \frac{\partial z}{\partial y} = -(-1)\sin(2x - y)$

$\frac{\partial z}{\partial y}\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)} = \sin\left(2\left(\frac{\pi}{4}\right) - \frac{\pi}{3}\right) = \frac{1}{2}$

Example 5:

#46 on the page 912

For the surface given by $f(x,y) = 3x^3 - 12xy + y^3$



to find the coordinates of all the points at which

$\frac{\partial f}{\partial x} = 0$ **and** $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} =$ **9x**^{2}**−12y**

$\frac{\partial f}{\partial y} =$ **−12x** $+$ **3y**^{2}

$\frac{\partial f}{\partial x} = 0$ **and** $\frac{\partial f}{\partial y} = 0$

implies that

$9x^2 - 12y = 0$ **and** $-12x + 3y^2 = 0$

9x^2**–12y = 0 → 12y = 9x**^{2}**→ y =** $\frac{9x^2}{12}$

substitute $y = \frac{3x^2}{4}$

in $-12x + 3y^2 = 0$

to obtain $-12x + 3\left(\frac{3x^2}{4}\right)^2 = 0$

which gives

–12x + $\frac{27x^4}{16}$ **= 0**

→

–4x + $\frac{9}{16}$**x**^{4}**= 0**

→

x$\left(\frac{9}{16}x^3 - 4\right)$**= 0**

→

$x = 0$ **or** $\frac{9}{16}x^3 - 4 = 0 \rightarrow 9x^3 = 64 \rightarrow x^3 = \frac{64}{9} \rightarrow x = \frac{4}{9^{1/3}} \rightarrow x = \frac{4}{3^{2/3}}$

substitute these values in

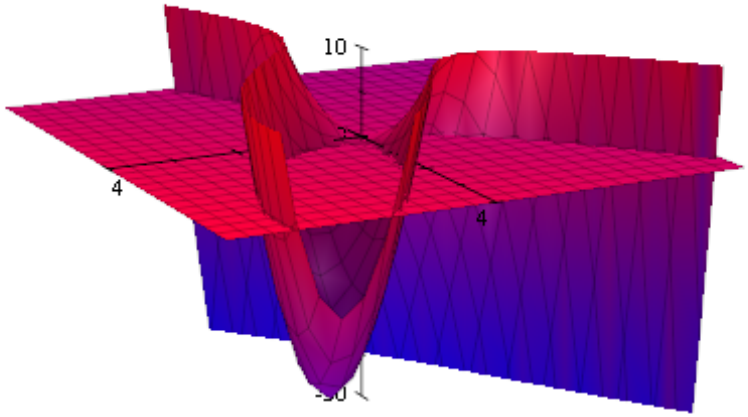
9x^2**–12y = 0**

x = 0 → y = 0

$\mathbf{x} = \frac{4}{3^{2/3}} \rightarrow \mathbf{9}\left(\frac{4}{3^{2/3}}\right)^2 - \mathbf{12y} = \mathbf{0} \rightarrow \mathbf{3}\left(\frac{16}{3^{4/3}}\right) - \mathbf{4y} = \mathbf{0} \rightarrow \frac{16}{3^{1/3}} - \mathbf{4y} = \mathbf{0} \rightarrow \mathbf{y} = \frac{4}{3^{1/3}}$

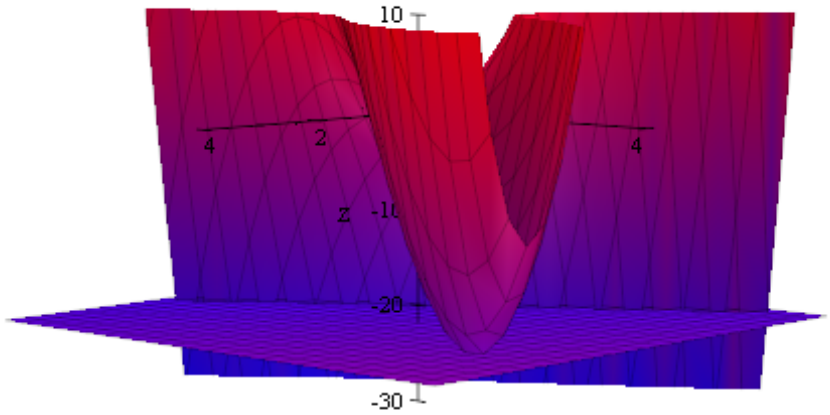
The coordinates of the required points are $(0,0)$ **and** $\left(\frac{16}{3^{1/3}}, \frac{4}{3^{1/3}}\right)$

at $(0,0)$ **we have**



The tangent plane is parallel to the xy-plane

at $\left(\frac{16}{3^{1/3}}, \frac{4}{3^{1/3}}\right)$



again the tangent plane is parallel to the xy-plane

Suggested Practice:

Section 13.3:

1 thru 97 odd numbered, 107 (must do)