Lesson4 Part3:

These notes correspond to the sections 13.1 and 13.2 in your text book.

Our text book has done a great job on the functions of several variables.

Out of many different texts that I have used for Multivariable Calculus, I have liked this one the best.

It will be very important that we read the text book.

Please make sure to read the pages 884-890 in the text before reading the lesson.

Let us look at some examples of functions of two variables:

Example 1:

$f(x,y) = 4 - x^2 - y^2$ **where both** x **and** y **are real numbers.**

Note that the values of $4 - x^2 - y^2$ **are available for all the values of** x **and** y

therefore the domain of this function is the set of all the values of (x,y) **in the x-y plane.**

The range of $f(x,y)$ **is the set of all values that** $z = f(x,y)$ **can assume for the values of**

(x,y) **in the domain.**

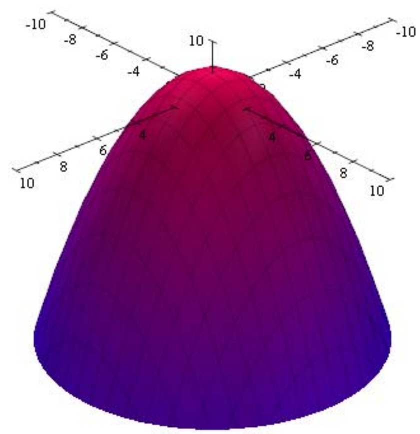
In this example, we have $f(x,y) = 4 - x^2 - y^2 = 4 - (x^2 + y^2)$

Note that Minimum that $x^2 + y^2$ can be is 0 therefore Maximum that $f(x,y)$ can be is 4

$x^2 + y^2$ **can be large without bounds, therefore** $f(x,y) = 4 - (x^2 + y^2)$ **will assume all possible values of 4 or less.**

i.e. the range of $f(x,y) = 4 - (x^2 + y^2)$ **is** $(-\infty, 4]$

If we look at a graph of $z = 4 - x^2 - y^2$**, it is**



Example 2:

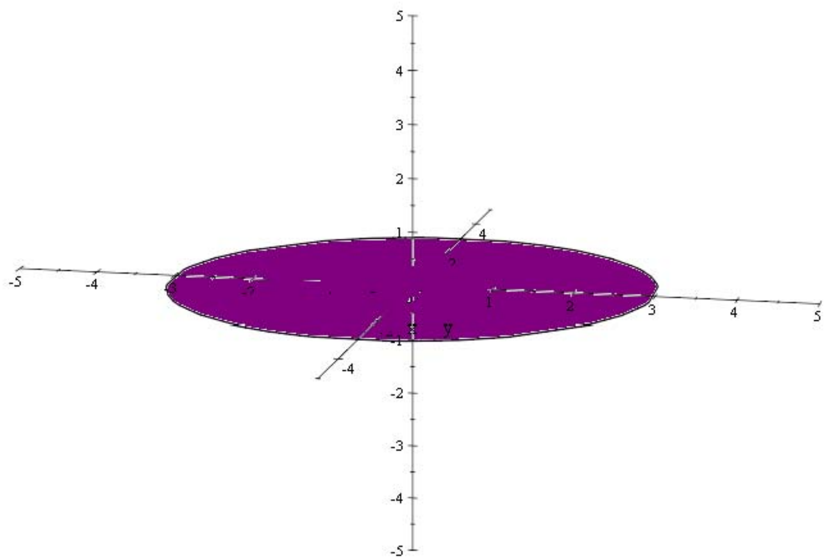
Consider $f(x,y) = \sqrt{9 - x^2 - y^2}$

Just think along the lines of the previous example, and note that a graph of

$z = \sqrt{9 - x^2 - y^2}$ **is the upper half of the sphere** $x^2 + y^2 + z^2 = 9$

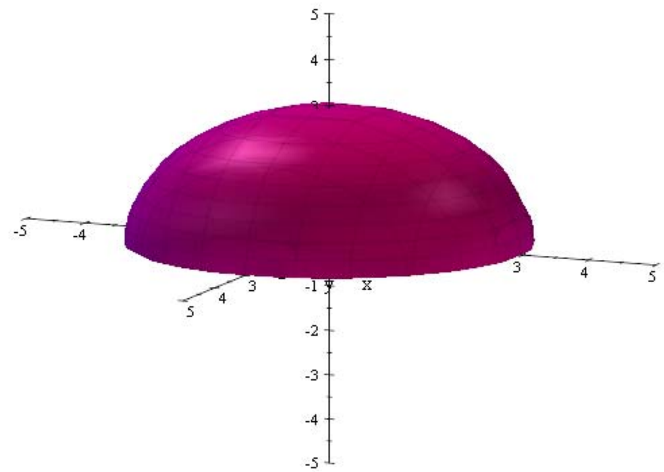
and the domain the the set $\{(x, y) : x^2 + y^2 \le 9\}$

that is



the range is $[0, 3]$

a graph is



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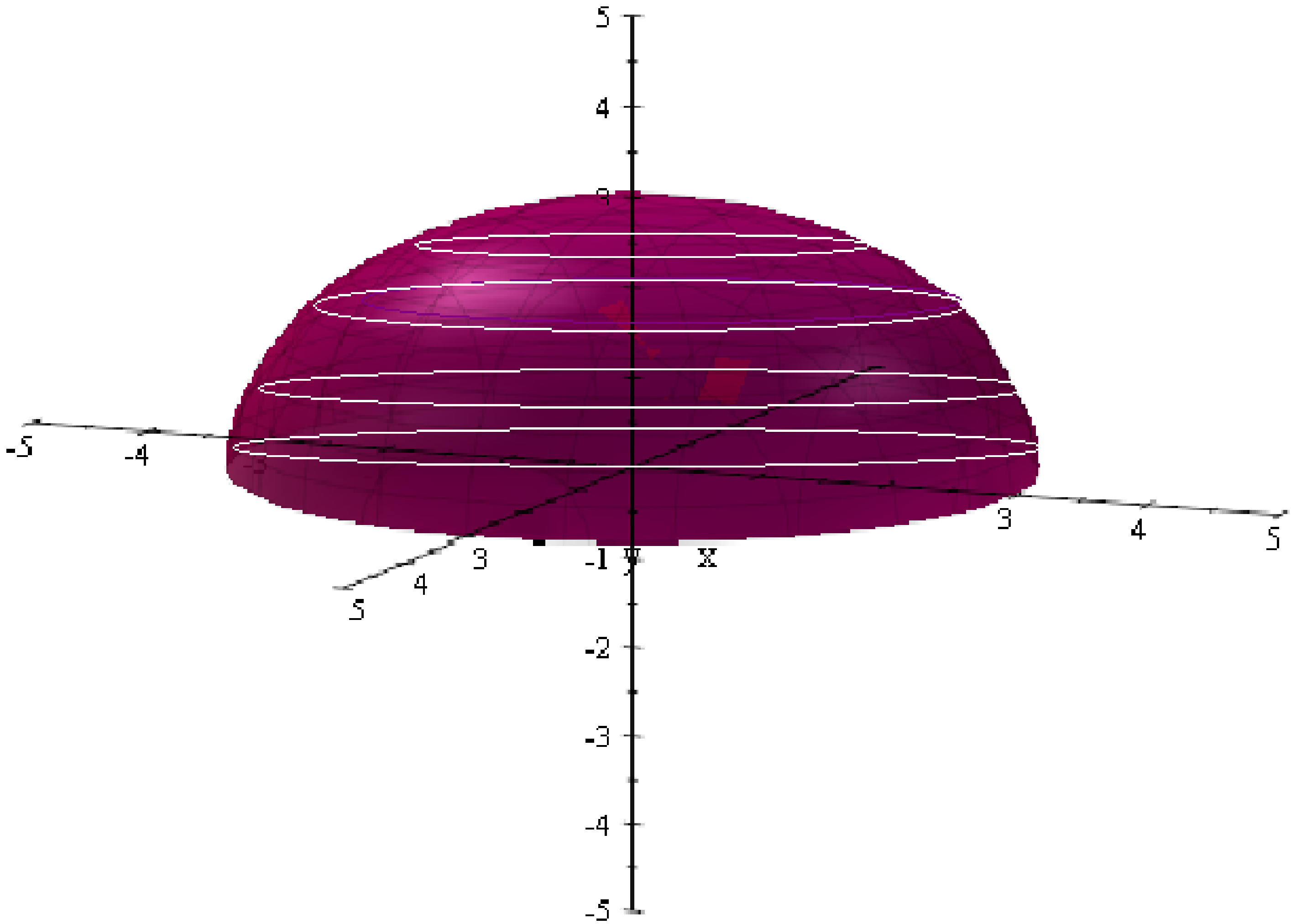


Note that for a function

$z = f(x,y)$**,** $f(x,y) = c$ **(constant) is a curve in the xy-plane which is called a level curve.**

for a function of three variables x, y, z

$w = f(x,y,z)$ **,** $f(x,y,z) = c$ **(constant) is a surface in the space, called a level surface.**

Example 3:

Let us look at #52 on the page 893

To describe the level curves of $f(x,y) = x^2 + 2y^2$ **, for** $c = 0, 2, 4, 6, 8$

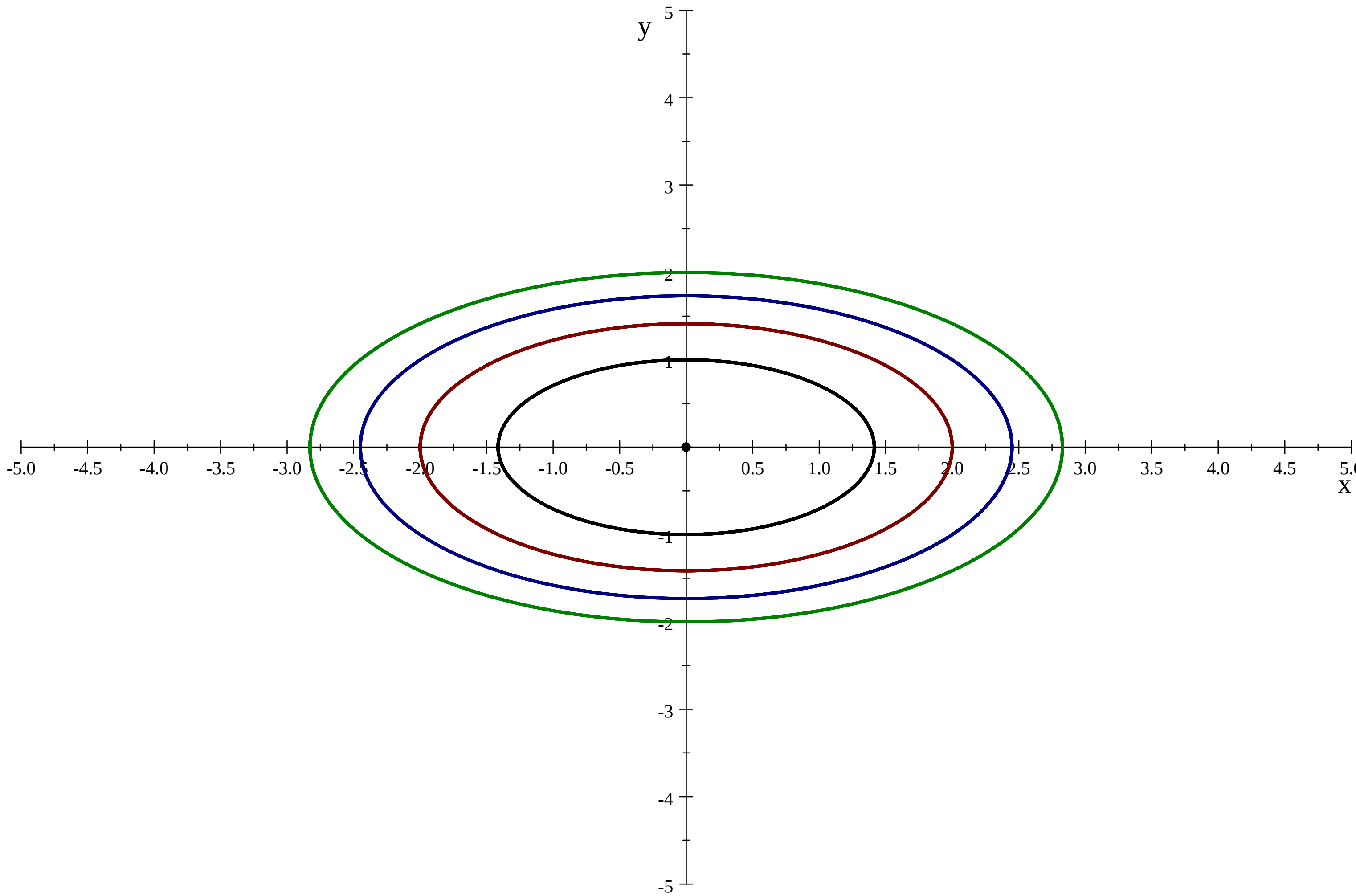
$x^2 + 2y^2 = 0$ **is just a point at the origin**

$x^2 + 2y^2 = 2$ **is the ellipse** $\frac{x^2}{2} + y^2 = 1$ **BLACK**

$x^2 + 2y^2 = 4$ **is the ellipse** $\frac{x^2}{4} + \frac{y^2}{2} = 1$ **RED**

$x^2 + 2y^2 = 6$ **is the ellipse** $\frac{x^2}{6} + \frac{y^2}{3} = 1$ **BLUE**

$x^2 + 2y^2 = 8$ **is the ellipse** $\frac{x^2}{8} + \frac{y^2}{4} = 1$ **GREEN**



Example 4:

Let us look at #54 on the page 493

To describe the level curves of $f(x,y) = e^{xy/2}$ **, for** $c = 2, 3, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

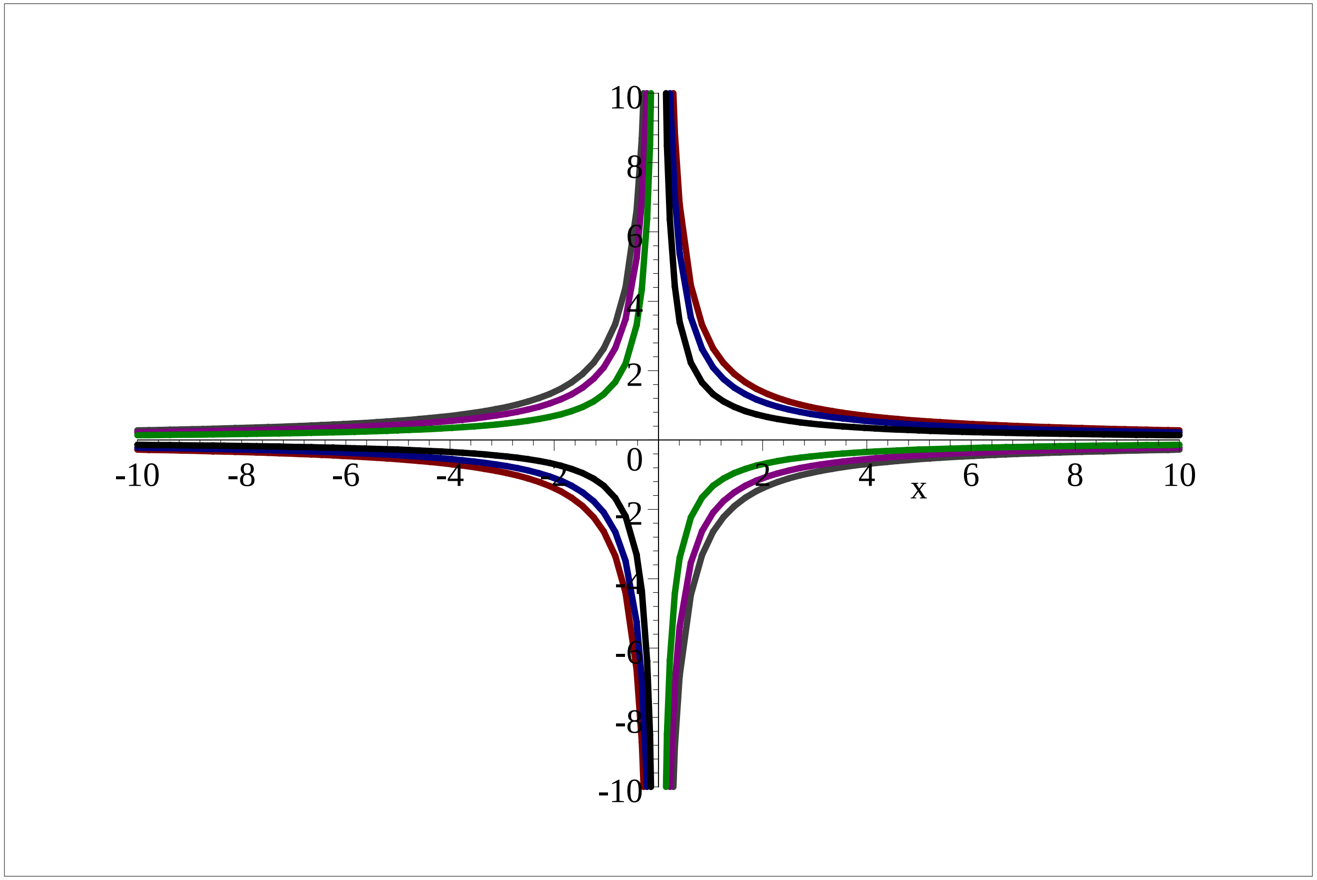
First, note that for any $c > 0$
The level curve is given by
$\mathbf{e}^{xy/2} = \mathbf{c}$

which gives us

$\ln(e^{xy/2}) = \ln \mathbf{c}$
$\rightarrow$
$\frac{xy}{2} = \ln \mathbf{c}$
$\rightarrow$
$\mathbf{y} = \frac{2 \ln c}{x}$

The level curve is going to be a rectangular hyperbola

$c = 2$ **(black)**
$c = 3$ **(blue)**
$c = 4$ **(red)**
$c = \frac{1}{2}$ **(green)**
$c = \frac{1}{3}$ **(purple)**
$c = \frac{1}{4}$ **(gray)**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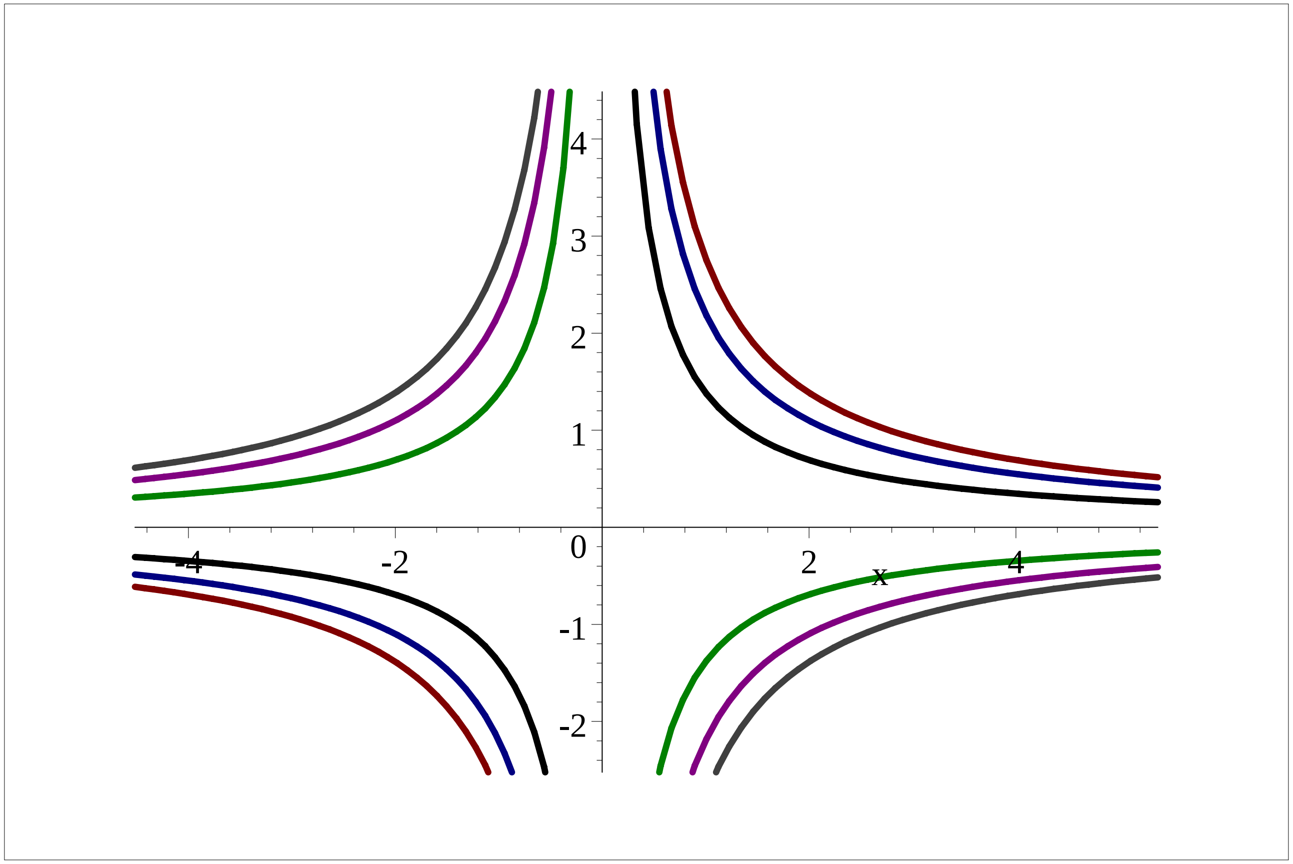
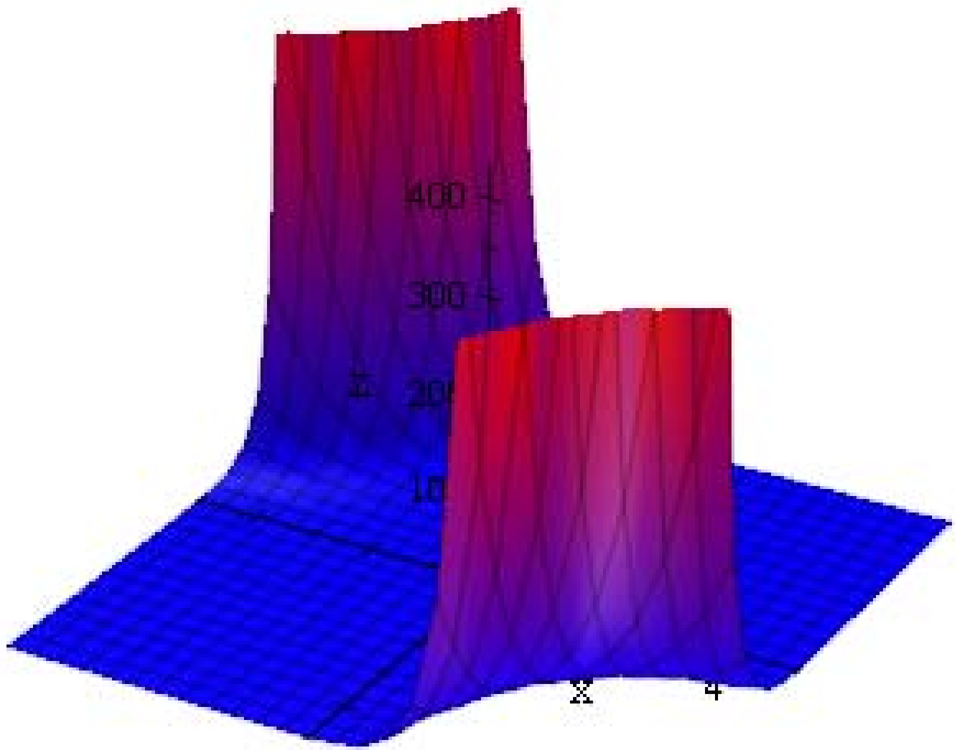


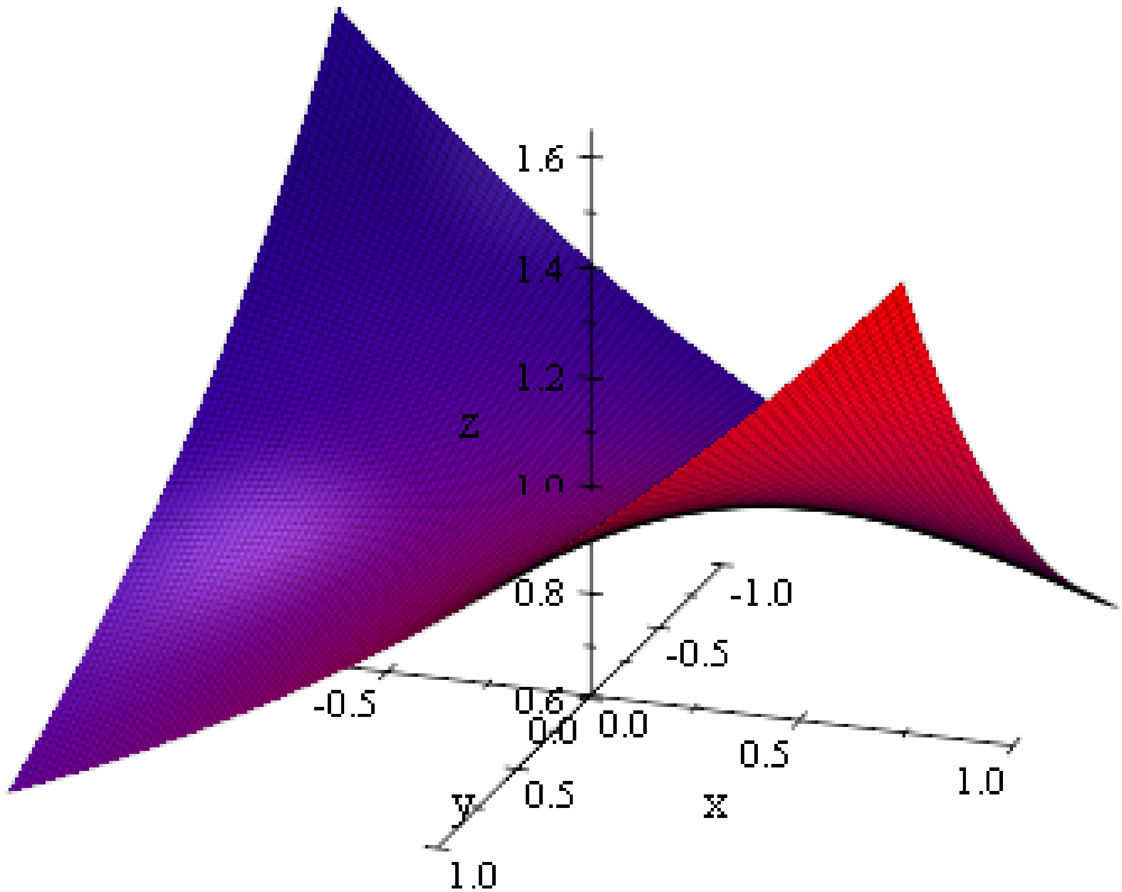
Note that the quadrants of the hyperbolae changed for $0 < c < 1$ because the natural log of such numbers is negative.

A graph of the surface

$z = e^{(xy/2)}$



if we zoom in close to the origin



Example 5:

Let us work on #72 page 894

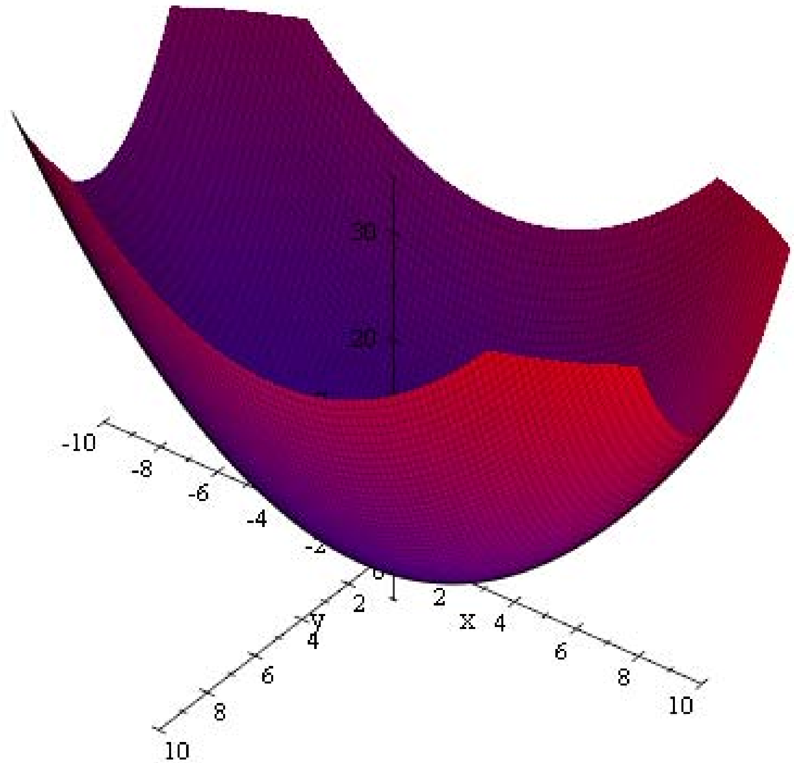
To sketch a level surface of

$f(x,y,z) = x^2 + \frac{1}{4}y^2 - z$**, for** $c = 1$

Note that

$\mathbf{x}^2+\frac{1}{4}\mathbf{y}^2-\mathbf{z} = \mathbf{1}$
$\rightarrow$
$\mathbf{z} = \mathbf{x}^2+\frac{1}{4}\mathbf{y}^2-\mathbf{1}$

This is a paraboloid with its vertex at $(0,0,-1)$ **as shown below**

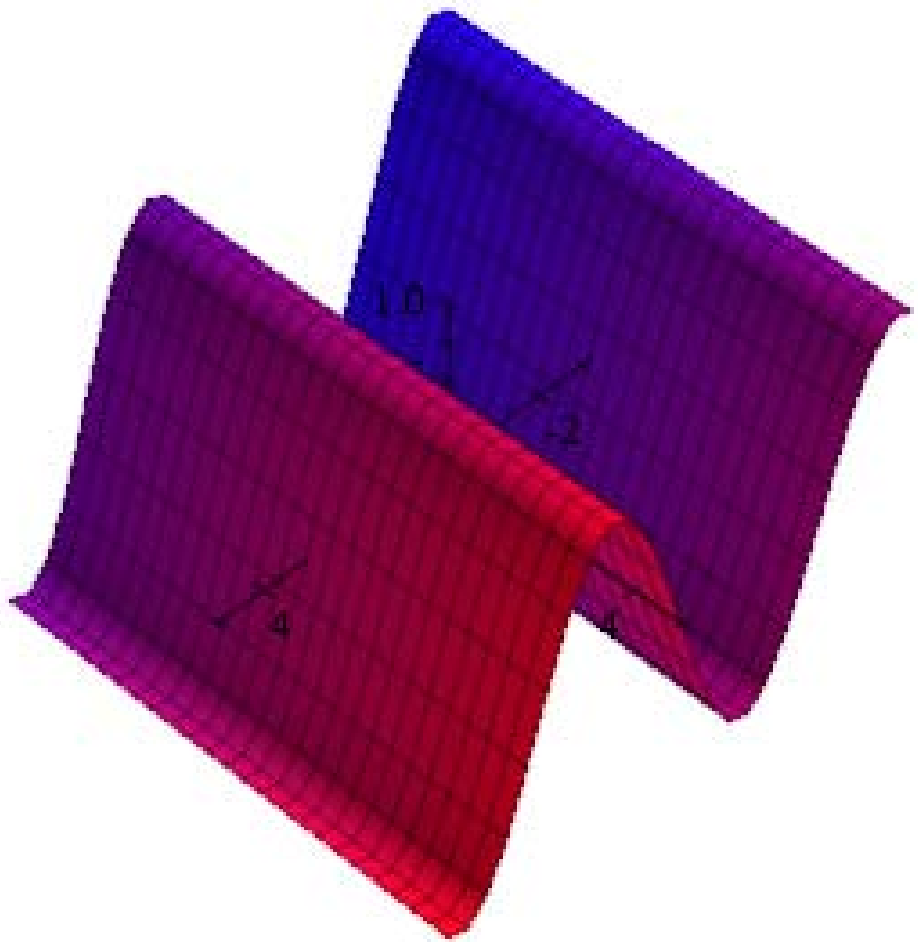


Example 6:

#74 page 894

To draw the level surface of $f(x, y, z) = \sin x - z$ **with** $c = 0$

Remember that graph of $\sin x - z = 0$ **that is** $z = \sin x$ **is a cylinder generated by the sine curve in the xz-plane with the rulings parallel to the y-axis**



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This part of the lesson corresponds with the section 13.2 in the text book

We must read the pages 896-902 of the text

Let us work on the evaluation of Limits:

Example 1:

#16 on the page 902

To evaluate $Lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2}$

Note that $xy \to 1$ **and** $x^2 + y^2 \to 1^2 + 1^2 = 2$

apply the algebra of limits to conclude that $Lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2} = \frac{1}{2}$

The domain of this function is the set of all points (x,y) **such that** $(x,y) \neq (0,0)$

and the function is continuous on its entire domain.

Example 2:

#22 on the page 903

To evaluate

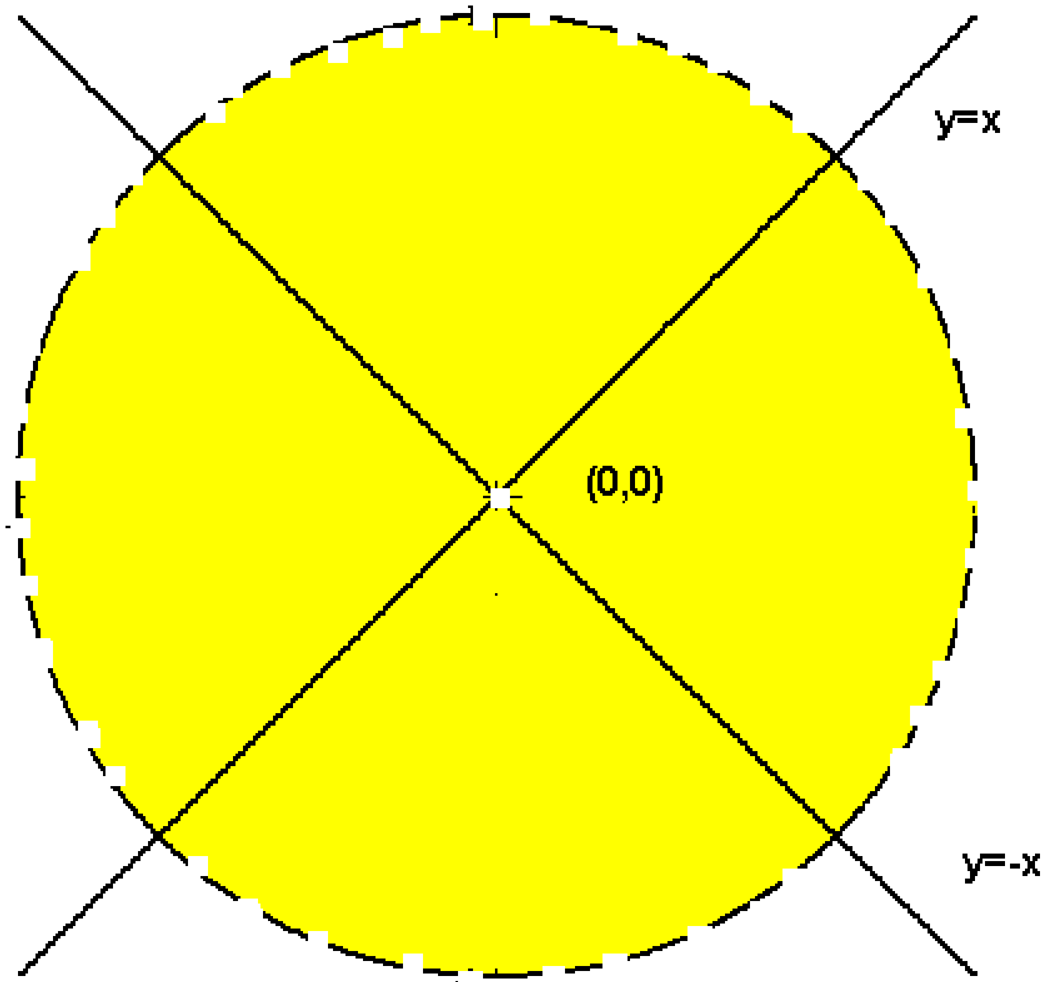
Lim$_{(x,y)\to(0,0)} \frac{x}{x^2-y^2}$

Note that the function $f(x,y) = \frac{x}{x^2-y^2}$ is not defined at the points (x,y) that lie on the line $y = x$ or $y = -x$

Look at the definition of the Limit on the page 297,

in order for the function $f(x,y)$ to have a limit at $(0,0)$, it must be defined in an open disk containing $(0,0)$ except possibly at $(0,0)$

But in this case, no matter which open disk we take, the function is not defined at any point that lie on the indicated lines



Therefore the function does not have a limit at $(0,0)$

Example 3:

#24 on the page 903

To find $Lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$

In this case the function $f(x,y,z) = \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$

is defined in any open sphere containing $(0,0,0)$ **except at** $(0,0,0)$

If we look at a table of values, see the Excel worksheet, it does not suggest the existence of a limit. I am posting another file with an embedded video of a descriptio of this spreadsheet.

To verify this statement, recall that in order for $f(x,y,z)$ **to have a limit L,**

the function should approach the value L regardless of the path the we adopt to approach $(0,0,0)$

If we approach $(0,0,0)$ **along the x-axis, then** $y = z = 0$ **along this path**

the function $f(x,y,z)$ **approaches** $\frac{x(0)+(0)(0)^2+(0)(0)^2}{x^2+(0)^2+(0)^2} = \frac{0}{x^2} = 0$

If we approach $(0,0,0)$ **along the line** $x = y = z$**, then along this path**

the function $f(x,y,z)$ **approaches** $\frac{x(x)+y(x)^2+x(x)^2}{x^2+(x)^2+(x)^2} = \frac{x^2+2x^3}{3x^2} = \frac{x^2(1+2x)}{3x^2} = \frac{1}{3}$

Since we get different values via different paths, the limit does not exist.

Example 4:

#44 on the page 905

To evaluate $Lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$

by using the polar coordinates as detailed in the text

x $=$ **r** $\cos\theta$ **y** $=$ **r** $\sin\theta$

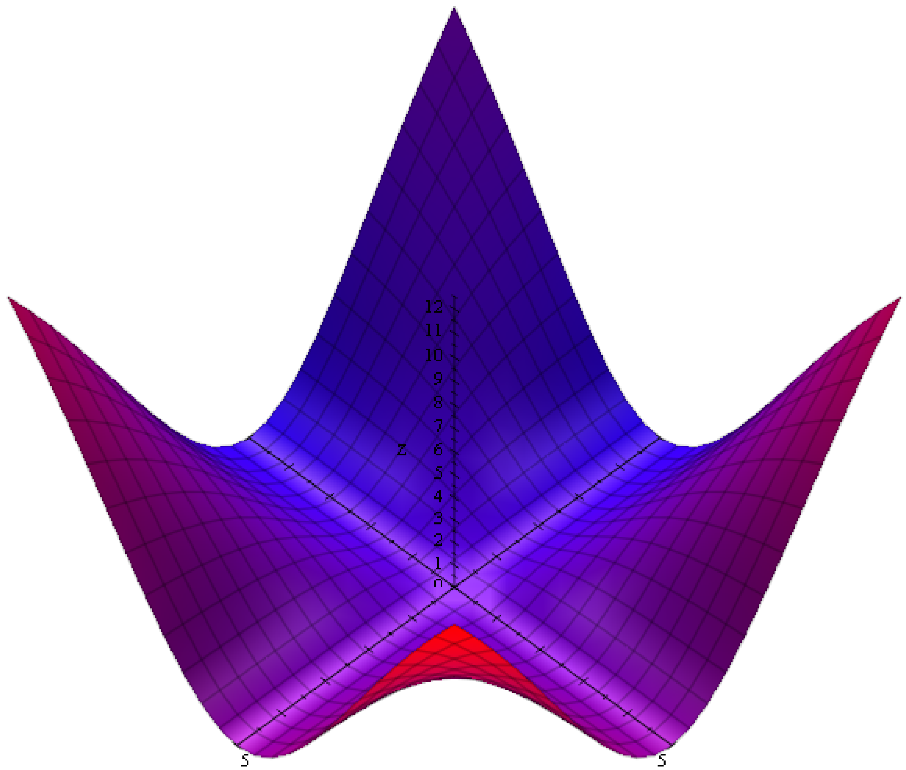
$$\mathbf{Lim}_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$$
$$= Lim_{(x,y)\to(0,0)} \frac{(r\cos\theta)^2(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}$$
$$= Lim_{(x,y)\to(0,0)} \frac{r^4\cos^2\theta\sin^2\theta}{r^2\left(\cos^2\theta+\sin^2\theta\right)}$$
$$= Lim_{(x,y)\to(0,0)} \frac{r^4\cos^2\theta\sin^2\theta}{r^2}$$
$$= Lim_{(x,y)\to(0,0)} r^2\cos^2\theta\sin^2\theta$$

since $r \to 0$ **as** $(x,y) \to (0,0)$

and $0 \le \cos^2\theta\sin^2\theta \le 1$

$$\mathbf{Lim}_{(x,y)\to(0,0)} \mathbf{r}^2\cos^2\boldsymbol{\theta}\sin^2\boldsymbol{\theta} = \mathbf{0}$$

$$Lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$$



Example 5:

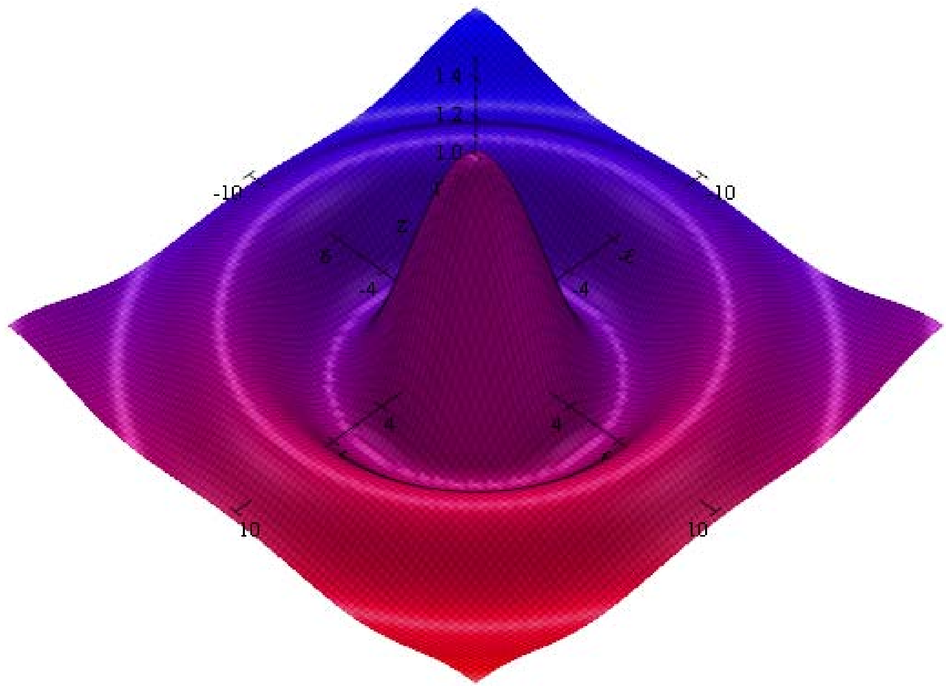
#46 in the text

To find $Lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{(x^2+y^2)}}{\sqrt{x^2+y^2}}$

substituting $x = r\cos\theta$ **and** $y = r\sin\theta$

we have

Lim$_{(x,y)\to(0,0)} \frac{\sin\sqrt{(x^2+y^2)}}{\sqrt{x^2+y^2}}$
$=Lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{r^2}}{\sqrt{r^2}}$
$=Lim_{(x,y)\to(0,0)} \frac{\sin r}{r}$ **if we have** $r > 0$
$=Lim_{r\to 0^+} \frac{\sin r}{r}$
=**1**



Example 7:

To discuss $Lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$

a)

Along the line $y = ax$

Lim$_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$

$=Lim_{(x,y)\to(0,0)}\frac{x^2(ax)}{x^4+(ax)^2}$

$=Lim_{(x,y)\to(0,0)}\frac{ax^3}{x^4+a^2x^2}$

$=Lim_{(x,y)\to(0,0)}\frac{ax^3}{x^2(x^2+a^2)}$

$=Lim_{(x,y)\to(0,0)}\frac{ax}{(x^2+a^2)}$

$=$**0**

b)

along $y = x^2$

$Lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$

$=Lim_{(x,y)\to(0,0)}\frac{x^2x^2}{x^4+(x^2)^2}$

$=Lim_{(x,y)\to(0,0)}\frac{x^4}{x^4+x^4}$

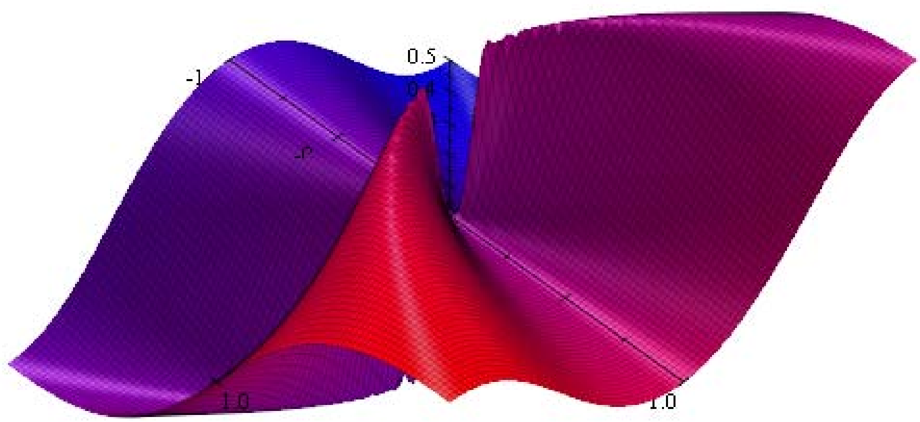
$=Lim_{(x,y)\to(0,0)}\frac{x^4}{2x^4}$

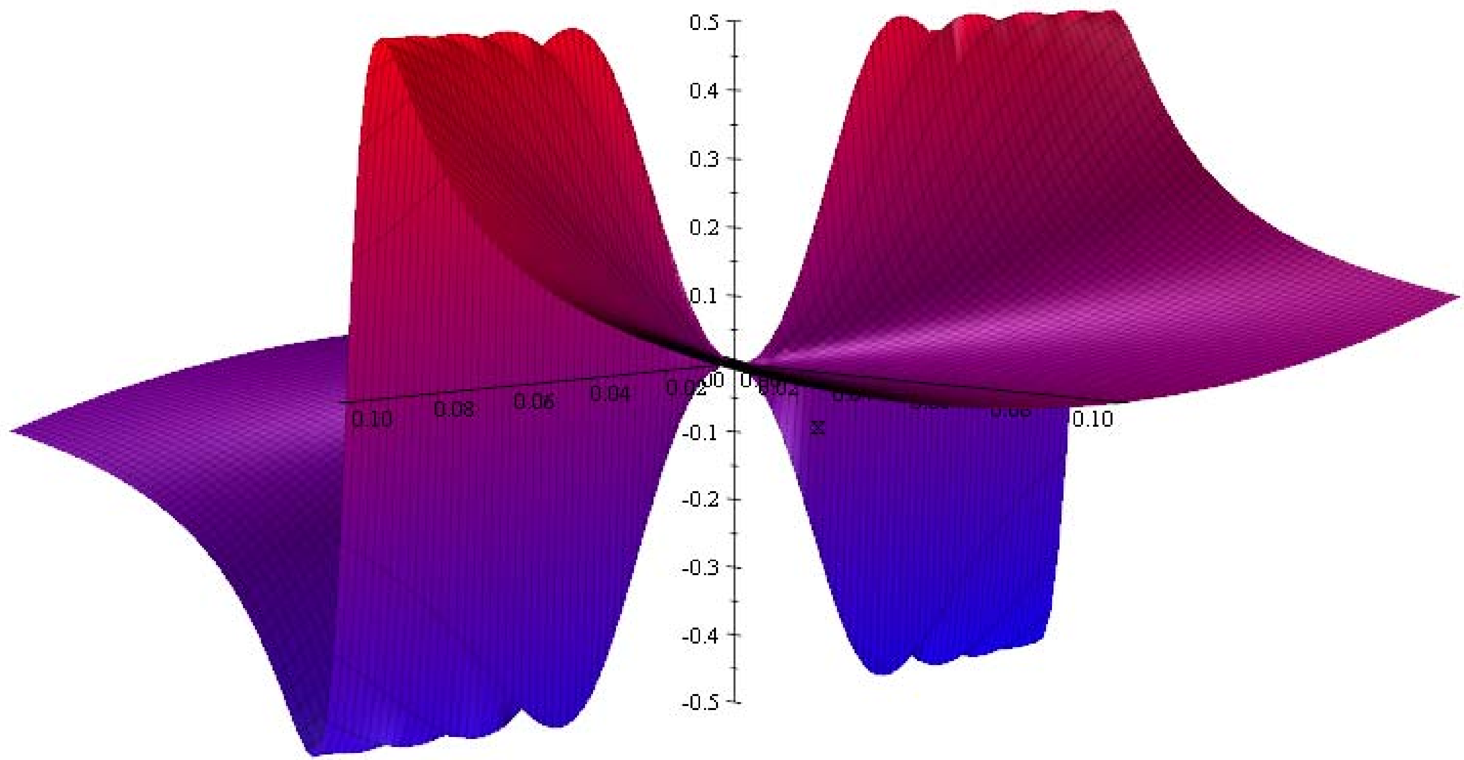
$=Lim_{(x,y)\to(0,0)}\frac{1}{2}$

c)

differrent paths yield different limits, therefore $Lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ **does not exist**

Look at diiferent views of some graphs of this function

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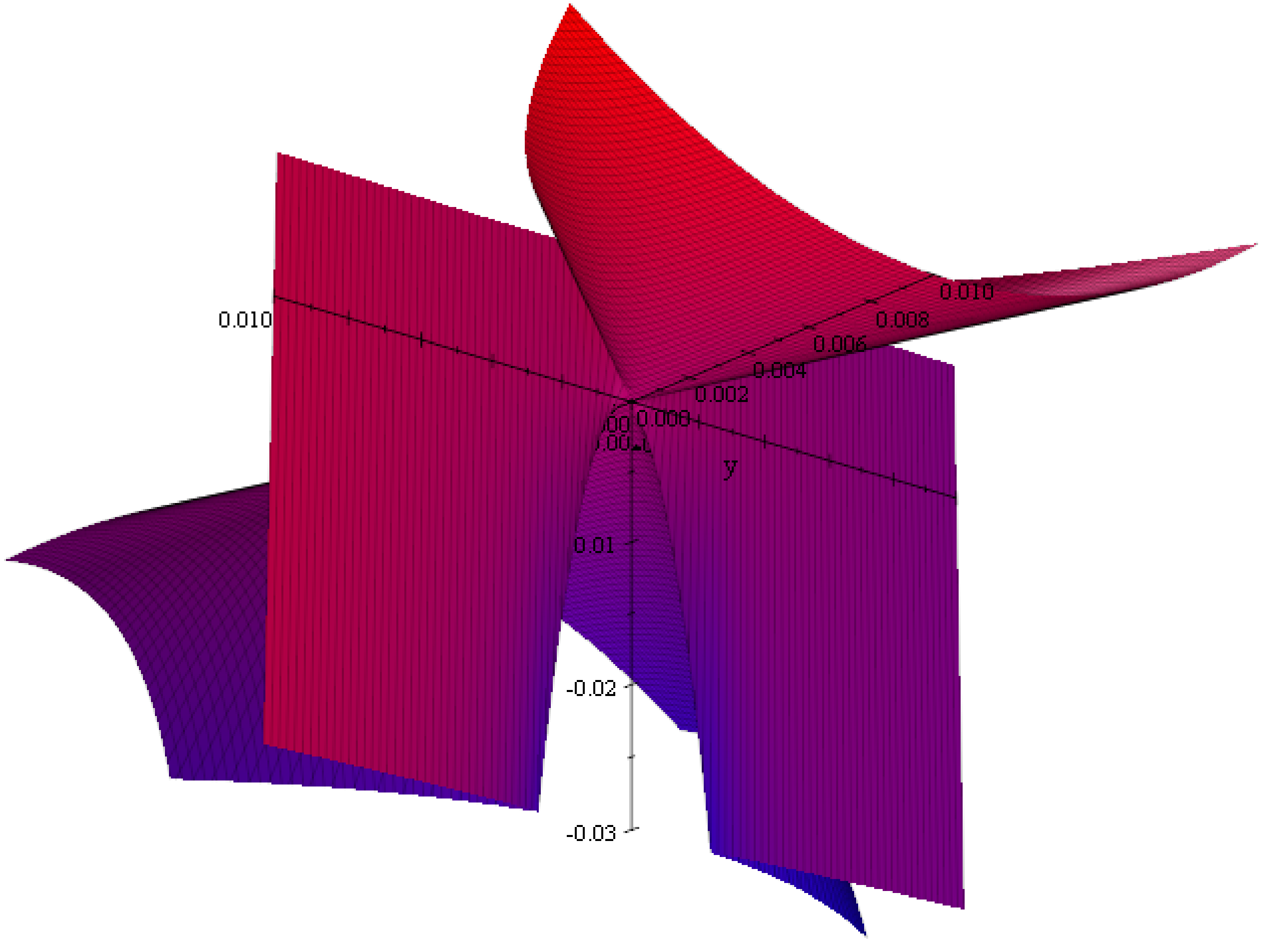

The graphs in this lesson were created by Scientific Work Place.

We discussed the procedure for this package earliar in week 2.

Suggested Practice:

Section 13.1: 7,11,13,19,23,27,35,47,49,51,55,67,69,71,77,83,87

Section 13.2: 1 thru 71 (odd numbered)