Lesson 3 Part 3

Remember that

$\vec{r}(t) = \mathbf{f(t)i} + \mathbf{g(t)j} + \mathbf{h(t)k}$

represnts a curve in the space.

Read the pages 849-851

and note that if $f(t), g(t), h(t)$ **are twice differentiable functions of** t

then

$\vec{v} = \frac{d\vec{r}}{dt}$ **and** $\vec{a} = \frac{d^2\vec{r}}{dt^2}$ **are the velocity at acceleration vectors at time** t

The speed is given by $||\vec{v}||$

Let us look at

exercise #16 on the page 854

To find the velocity, speed and the acceleration of the object that travels along
$\vec{r} = \langle e^t \cos t, e^t \sin t, e^t \rangle$

First note that

$\mathbf{x} = \mathbf{e}^t \cos \mathbf{t}$

$y = e^t \sin t$
$z = e^t$

gives

$x^2+y^2= e^{2t}\cos^2 t + e^{2t}\sin^2 t$

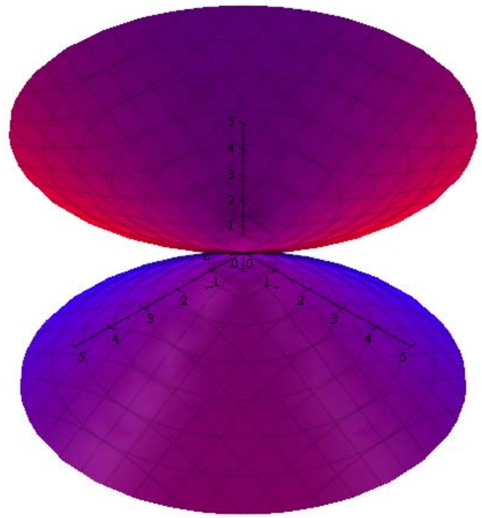
or

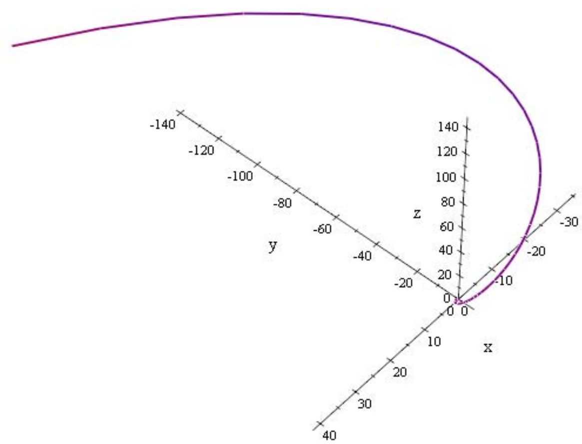
$x^2+y^2= e^{2t}\left(\cos^2 t + \sin^2 t\right)$

or

$x^2+y^2= z^2$

which means that the curve is contained on the cone





The velocity vector

$$\vec{v}=\frac{d\vec{r}}{dt}=\frac{d}{dt}[(e^t\cos t)i+(e^t\sin t)j+e^tk]=(e^t\cos t-e^t\sin t)\mathbf{i}+(e^t\sin t+e^t\cos t)\mathbf{j}+\mathbf{e}^t\mathbf{k}$$

$$\vec{v}=(e^t\cos t-e^t\sin t)\mathbf{i}+(e^t\sin t+e^t\cos t)\mathbf{j}+\mathbf{e}^t\mathbf{k}$$

$$||\vec{v}||$$
$$=\sqrt{(e^t\cos t-e^t\sin t)^2+((e^t\sin t+e^t\cos t))^2+(e^t)^2}$$
$$=\sqrt{e^{2t}\cos^2 t+e^{2t}\sin^2 t-2e^{2t}\cos t\sin t+e^{2t}\cos^2 t+e^{2t}\sin^2 t+2e^{2t}\cos t\sin t+e^{2t}}$$
$$=\sqrt{e^{2t}\cos^2 t+e^{2t}\sin^2 t+e^{2t}\cos^2 t+e^{2t}\sin^2 t+e^{2t}}$$
$$=\sqrt{e^{2t}\left(\cos^2 t+\sin^2 t\right)+e^{2t}\left(\cos^2 t+\sin^2 t\right)+e^{2t}}$$

$=\sqrt{e^{2t}+e^{2t}+e^{2t}}$
$=\sqrt{3e^{2t}}$
$=e^{t}\sqrt{3}$

The acceleration vector is

$$\vec{a}=\frac{d\vec{v}}{dt}=(e^t\cos t-e^t\sin t-e^t\sin t-e^t\cos t)\mathbf{i}+(e^t\sin t+e^t\cos t+e^t\cos t-e^t\sin t)\mathbf{j}+\mathbf{e}^t\mathbf{k}$$

$$\vec{a}=-(e^t\sin t)\mathbf{i}+(e^t\cos t)\mathbf{j}+\mathbf{e}^t\mathbf{k}$$

APPLICATION:

Let us consider the following motion in a plane with the path described by the position vector

$$\vec{r}=\mathbf{5}\cos\left(\tfrac{1}{4}t\right)\mathbf{i}+\mathbf{5}\sin\left(\tfrac{1}{4}t\right)\mathbf{j}$$

Note that

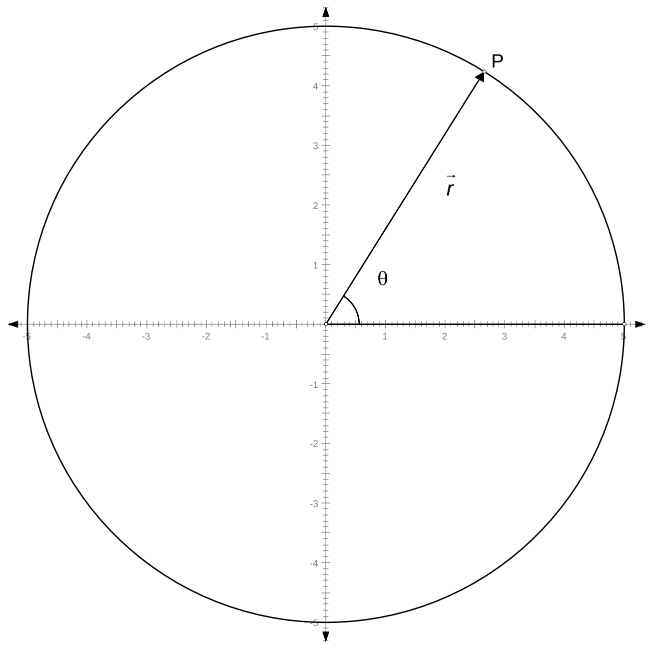
$\mathbf{x}=\mathbf{5}\cos\left(\tfrac{1}{4}t\right)$ $\qquad$ $\mathbf{y}=\mathbf{5}\sin\left(\tfrac{1}{4}t\right)$

$$\mathbf{x}^2+\mathbf{y}^2=\mathbf{25}\cos^2\left(\tfrac{1}{4}t\right)+\mathbf{25}\sin^2\left(\tfrac{1}{4}t\right)$$

$$\mathbf{x}^2+\mathbf{y}^2=\mathbf{25}\left(\cos^2\left(\tfrac{1}{4}t\right)+\sin^2\left(\tfrac{1}{4}t\right)\right)$$

$$\mathbf{x}^2+\mathbf{y}^2=\mathbf{25}$$

The path is slong the circle



Where $\theta = \frac{1}{4}t$

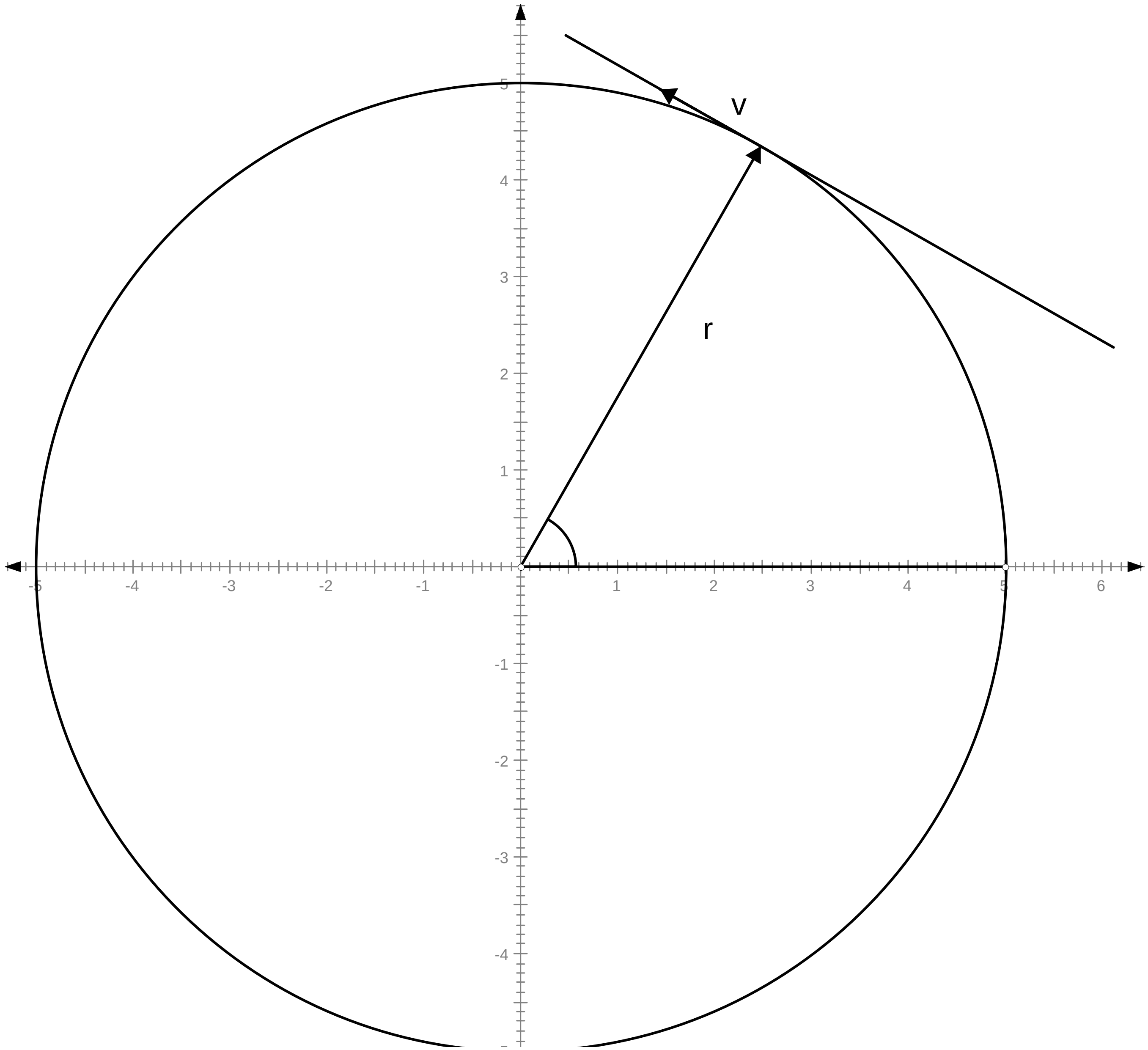
Note that $\frac{d\theta}{dt} = \frac{1}{4}$ **rad/per unit time**

Note that

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(5\cos\left(\frac{1}{4}t\right)i + 5\sin\left(\frac{1}{4}t\right)j\right) = -\left(\frac{5}{4}\sin\left(\frac{1}{4}t\right)\right)\mathbf{i} + \left(5\cos\left(\frac{1}{4}t\right)\right)\mathbf{j}$$

You can easily note that

$\vec{r} \cdot \vec{v} =$ **0**


v
r
5
4
3
2
1
-1
-2
-3
-4
-5
-4
-3
-2
-1
1
2
3
4
5
6

The acceleration vector is $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(-\left(\frac{5}{4}\sin\left(\frac{1}{4}t\right)\right)i + \left(5\cos\left(\frac{1}{4}t\right)\right)j\right)$

$\vec{a} = -\frac{5}{16}\left(\cos\left(\frac{1}{4}t\right)\right)\mathbf{i} - \frac{5}{16}\left(\sin\left(\frac{1}{4}t\right)\right)\mathbf{j}$

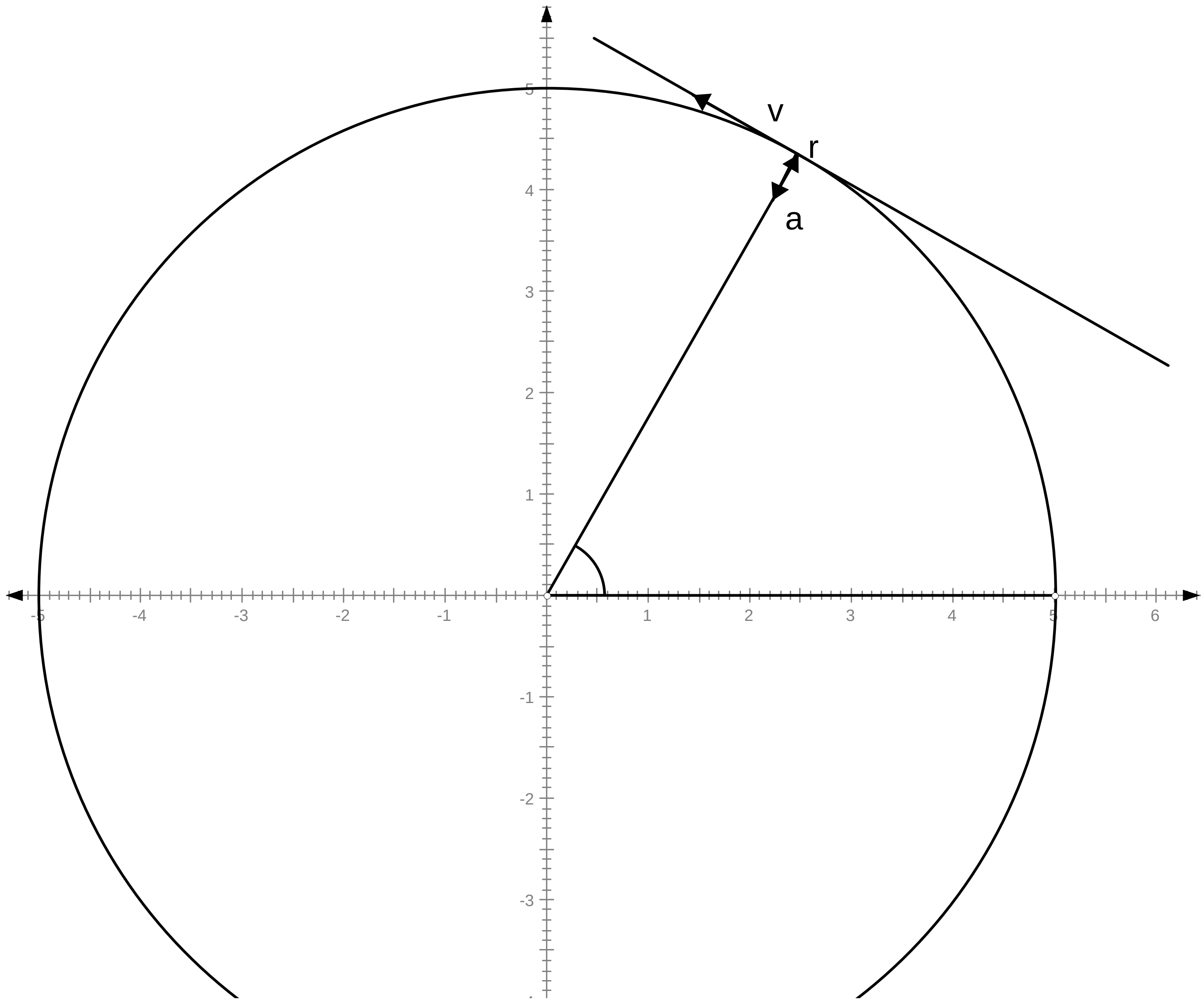
Note that $||\vec{a}|| = \frac{5}{16}$

and also

$\vec{a} = -\frac{1}{16}\left(5\left(\cos\left(\frac{1}{4}t\right)\right)i + 5\left(\sin\left(\frac{1}{4}t\right)\right)j\right)$

or

$\vec{a} = -\frac{1}{16}\vec{r}$


v
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a
5
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-5
-4
-3
-2
-1
1
2
3
4
5
6

In general if a particle moves along a circle given by $\vec{r} = b\cos(wt)i + b\sin(wt)j$

then

$\vec{v} = \frac{d\vec{r}}{dt} = -\mathbf{wb}((\sin(wt))i + wb(\cos(wt))j)$

$\vec{v} = bw\ (-(\sin(wt))i + (\cos(wt))j)$

$||\vec{v}|| = |bw|$

$\vec{a} = \frac{d\vec{v}}{dt}$

$\vec{a} = -\mathbf{bw}^2(\cos(wt)i + \sin(wt)j)$

$\vec{a} = -\mathbf{w}^2\ \vec{r}$

....

On the other hand we know the acceleration vector, $\frac{d^2\vec{r}}{dt^2}$

we can integrate twice and find the position vector function $\vec{r}(t)$

Let us take up

#22 on the page 854

Given that

$\vec{a} = -(\cos t)i - (\sin t)j$ **GIVEN THAT** $\vec{v}(0) = j + k$ $\vec{r}(0) = i$

$\frac{d\vec{v}}{dt} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$

Integrate with respect to t

$\vec{v} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \vec{C}$

Given that $\vec{v}(0) = j + k$

Therefore

$-(\sin 0)\mathbf{i} + (\cos 0)\mathbf{j} + \vec{C} = \mathbf{j} + \mathbf{k}$
$\mathbf{j} + \vec{C} = \mathbf{j} + \mathbf{k}$

Therefore $\vec{C} = k$

$\vec{v} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \vec{C}$
$\rightarrow$
$\frac{d\vec{r}}{dt} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$

integrate with respect to t

$\vec{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \mathbf{tk} + \vec{D}$

Given that

$$\vec{r}(0) = \mathbf{i}$$

$$(\cos 0)\mathbf{i} + (\sin 0)\mathbf{j} + (0)\mathbf{k} + \vec{D} = \mathbf{i}$$

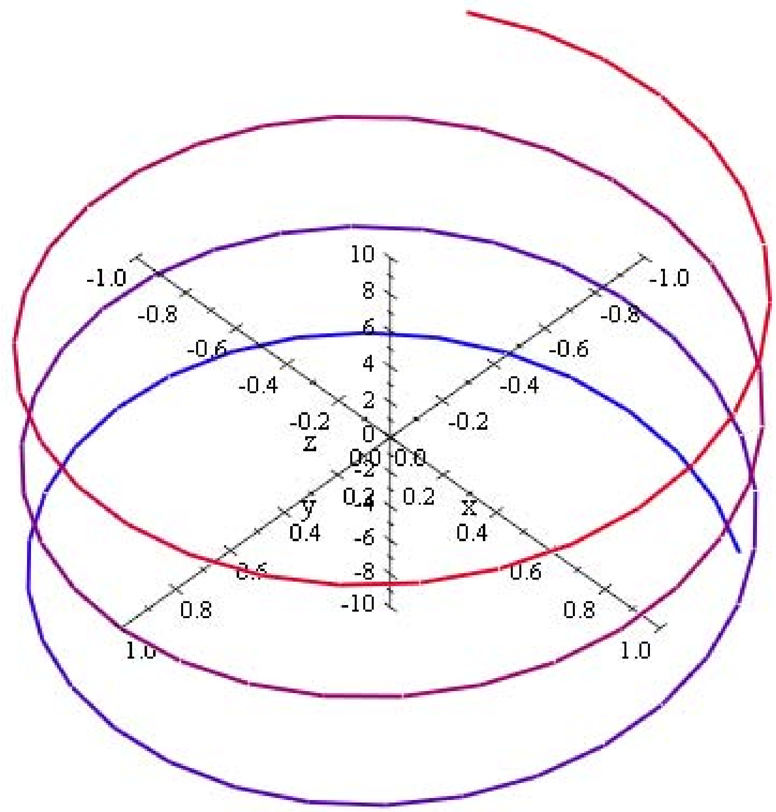
$$\mathbf{i} + \vec{D} = \mathbf{i}$$

$$\rightarrow$$

$$\vec{D} = \mathbf{0}$$

$$\vec{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \mathbf{2tj} + \mathbf{tk}$$

$$\vec{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \mathbf{tk}$$


10
8
6
4
2
0
-2
-4
-6
-8
-10
z
y
x
-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0

A nice example of obtaining the position vector function $\vec{r}$ by integrating $\vec{a}$ twice

is the example where

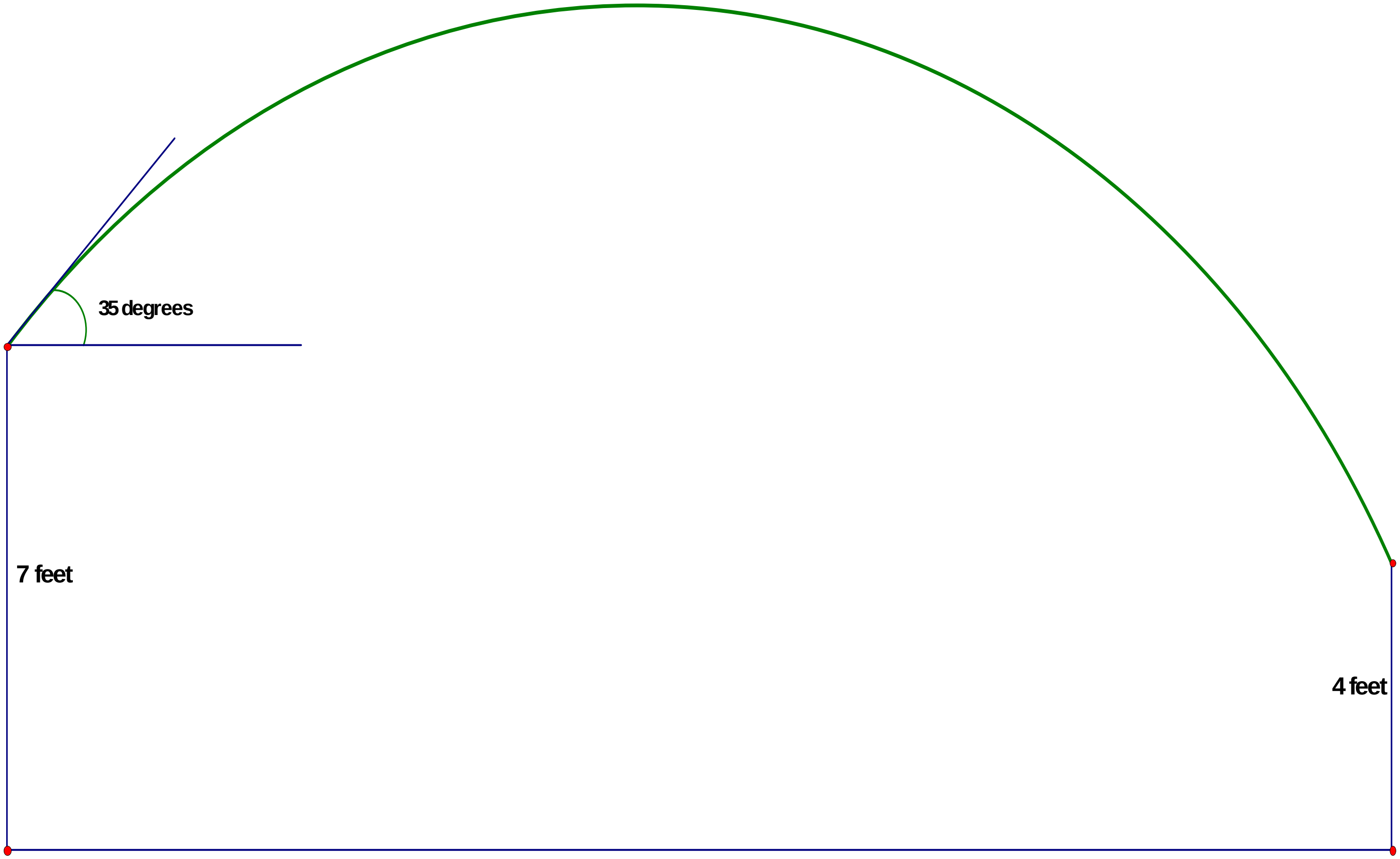
$\vec{a} = -g\mathbf{j}$

where g is the acceleration due to gravity, is described on the pages 852-853 of the text

I am going to show you the steps using an exercise in the text book:

#34 on the page 855

Even though you can just use the summary formulas, let us still work on it from scratch.



Considering this as a motion in the i-j plane

Note that the acceleration vector is

$$\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = -\mathbf{g}\ \mathbf{j}$$

where $g = 32\ ft/\sec^2$

$\frac{d\vec{v}}{dt} = -\mathbf{g}\ \mathbf{j}$

$\rightarrow$

$\vec{v} = -\mathbf{gt}\ \mathbf{j} + \vec{C}$

If the speed at the time of release is $v_\circ$ **feet per second**

the inital velocity $\vec{v_\circ}$ **as a vector is**

$\vec{v_\circ} = \mathbf{v}_\circ(\cos 35^\circ)\mathbf{i} + \mathbf{v}_\circ(\sin 35^\circ)\mathbf{j}$

that is, when $t = 0$**,** $\vec{v} = \vec{v_\circ} = (\cos 35^\circ)i + (\sin 35^\circ)j$

$\mathbf{v}_\circ(\cos 35^\circ)\mathbf{i} + \mathbf{v}_\circ(\sin 35^\circ)\mathbf{j} = -\mathbf{g}(0)\ \mathbf{j} + \vec{C}$

therefore

$\vec{C} = \mathbf{v}_\circ(\cos 35^\circ)\mathbf{i} + \mathbf{v}_\circ(\sin 35^\circ)\mathbf{j}$

$\vec{v} = -\mathbf{gt}\ \mathbf{j} + \mathbf{v}_\circ(\cos 35^\circ)\mathbf{i} + \mathbf{v}_\circ(\sin 35^\circ)\mathbf{j}$
$\vec{v} = \mathbf{v}_\circ(\cos 35^\circ)\mathbf{i} + (-gt + v_\circ \sin 35^\circ)\mathbf{j}$

$\frac{d\vec{r}}{dt} = \mathbf{v}_\circ(\cos 35^\circ)\mathbf{i} + (-gt + v_\circ \sin 35^\circ)\mathbf{j}$

Integrate

$\vec{r} = \mathbf{v_\circ t}(\cos 35^\circ)\mathbf{i} + \left(-g\frac{t^2}{2} + v_\circ t\sin 35^\circ\right)\mathbf{j} + \vec{D}$

Since, the ball was project initially from a height of 7 feet, taking the position of the Quarter back as the origin, the position of the ball at time $t = 0$ sec is

$\vec{r_\circ} =$ **7j**

When $t = 0$, we have $\vec{r} = \vec{r_\circ} = 7j$

7j $= \mathbf{v_\circ}(0)(\cos 35^\circ)\mathbf{i} + \left(-g\frac{0^2}{2} + v_\circ(0)\sin 35^\circ\right)\mathbf{j} + \vec{D}$

$\vec{D} =$ **7j**

$\vec{r} = \mathbf{v_\circ t}(\cos 35^\circ)\mathbf{i} + \left(-g\frac{t^2}{2} + v_\circ t\sin 35^\circ\right)\mathbf{j} + \mathbf{7j}$

$\vec{r} = \mathbf{v_\circ t}(\cos 35^\circ)\mathbf{i} + \left(7 - g\frac{t^2}{2} + v_\circ t\sin 35^\circ\right)\mathbf{j}$

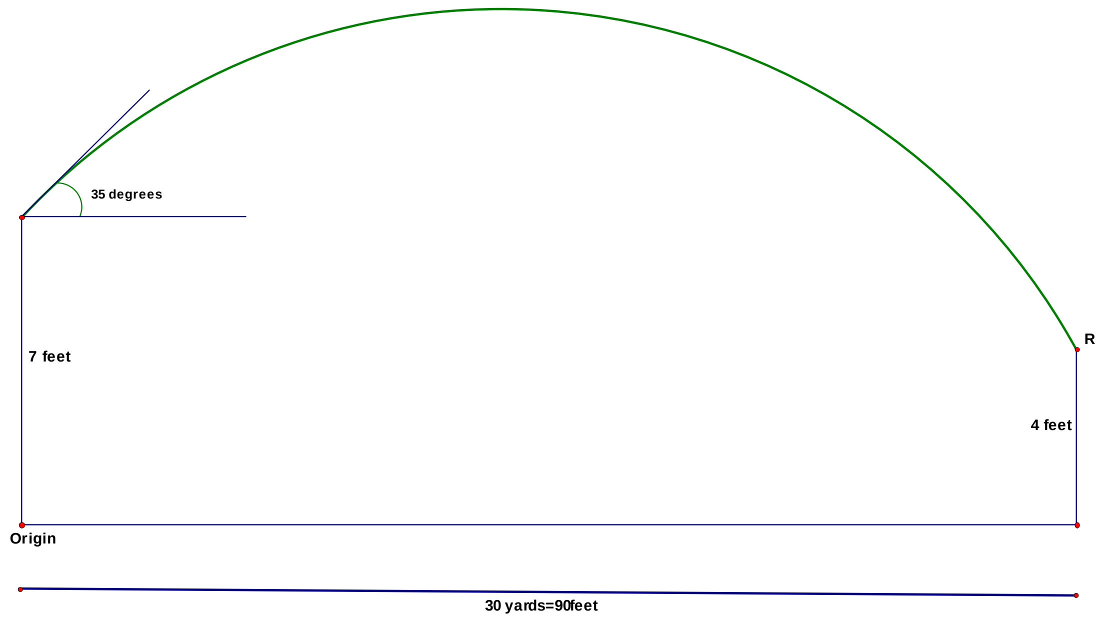
Compare this with the equation

$\boxed{\vec{r}(t) = v_\circ t(\cos\theta)i + \left(h - g\frac{t^2}{2} + v_\circ t\sin\theta\right)j}$ **on the page 853 of the text book**

let us work on the problems now,

We have $x = v_\circ t(\cos 35^\circ)$ **and** $y = \left(7 - g\frac{t^2}{2} + v_\circ t\sin 35^\circ\right)$

a)



Note that (300,4) is a point on the curve that is given by

$x = v_\circ t(\cos 35^\circ)$ **and** $y = \left(7 - g\frac{t^2}{2} + v_\circ t \sin 35^\circ\right)$

At the point, where the ball is received

$v_\circ t(\cos 35^\circ) = 90$ **.......... (1)**

$$\left(7 - g\frac{t^2}{2} + v_\circ t\sin 35^\circ\right) = 4$$ **.......... (2)**

Equation (1) gives

$t = \frac{90}{v_\circ(\cos 35^\circ)}$ **at R**

Substitute the value is (2)

$$\left(7 - 32 \times \frac{\left(\frac{90}{v_\circ(\cos 35^\circ)}\right)^2}{2} + v_\circ\frac{90}{v_\circ(\cos 35^\circ)}\sin 35^\circ\right) = 4$$

$\rightarrow$

$$\left(7 - 32 \times \frac{\left(\frac{90}{v_\circ(\cos 35^\circ)}\right)^2}{2} + \frac{90}{(\cos 35^\circ)}\sin 35^\circ\right) = 4$$

$\rightarrow$

$$\left(7 - 16\left(\frac{90}{v_\circ(\cos 35^\circ)}\right)^2 + \frac{90}{(\cos 35^\circ)}\sin 35^\circ\right) = 4$$

$\rightarrow$

$$-\mathbf{16}\left(\frac{90}{v_\circ(\cos 35^\circ)}\right)^2 = \mathbf{4} - \mathbf{7} - \frac{90}{(\cos 35^\circ)}\sin \mathbf{35}^\circ$$

$\rightarrow$

$$\left(\frac{90}{v_\circ(\cos 35^\circ)}\right)^2=\left(\frac{-3-\frac{90}{(\cos 35^\circ)}\sin 35^\circ}{-16}\right)$$

→

$$\left(\frac{v_\circ(\cos 35^\circ)}{90}\right)^2=\left(\frac{16}{3+\frac{90}{(\cos 35^\circ)}\sin 35^\circ}\right)$$

→

$$\mathbf{v}_\circ^2=\left(\frac{16}{3+90\tan 35^\circ}\right)\left(\frac{900}{\cos 35^\circ}\right)^2$$

→

$$v_\circ = \sqrt{\left(\frac{16}{3+90\tan 35^\circ}\right)}\left(\frac{90}{\cos 35^\circ}\right)\cong 54.088 \textbf{ feet/sec}$$

b) The height is given by

$$\mathbf{y} = \left(7-g\frac{t^2}{2}+v_\circ t\sin 35^\circ\right)=\left(7-16t^2+t(\sin 35^\circ)\sqrt{\left(\frac{16}{3+90\tan 35^\circ}\right)}\left(\frac{90}{\cos 35^\circ}\right)\right)$$

The max occurs at

$$t = \frac{(\sin 35^\circ)\sqrt{\left(\frac{16}{3+90\tan 35^\circ}\right)}\left(\frac{90}{\cos 35^\circ}\right)}{32}\cong 0.9695 \textbf{ sec}$$

Therefore, the max height is

$$y = \left(7 - 16\left(\frac{(\sin 35^\circ)\sqrt{\left(\frac{16}{3+90\tan 35^\circ}\right)}\left(\frac{90}{\cos 35^\circ}\right)}{32}\right)^2 + \left(\sqrt{\left(\frac{16}{3+90\tan 35^\circ}\right)}\left(\frac{90}{\cos 35^\circ}\right)\right)\left(\frac{(\sin 35^\circ)\sqrt{\left(\frac{16}{3+90\tan 35^\circ}\right)}\left(\frac{90}{\cos 35^\circ}\right)}{32}\right)\sin 35^\circ\right) \cong 22.038750873585077962$$ **feet**

c)

The time that it takes the football to reach the position R is (as calculated in part a)

t $= \frac{90}{v_\circ(\cos 35^\circ)}$

or

$$t = \frac{90}{\sqrt{\left(\frac{16}{3+90\tan 35^\circ}\right)}\left(\frac{90}{\cos 35^\circ}\right)(\cos 35^\circ)} \cong 2.0312969754394893771$$ **sec**

Section 12.3: 3,5,7,11,13,15,17,19,21,27,29,31,35,41,45,49