Section 12.1
Please read the pages 832-836

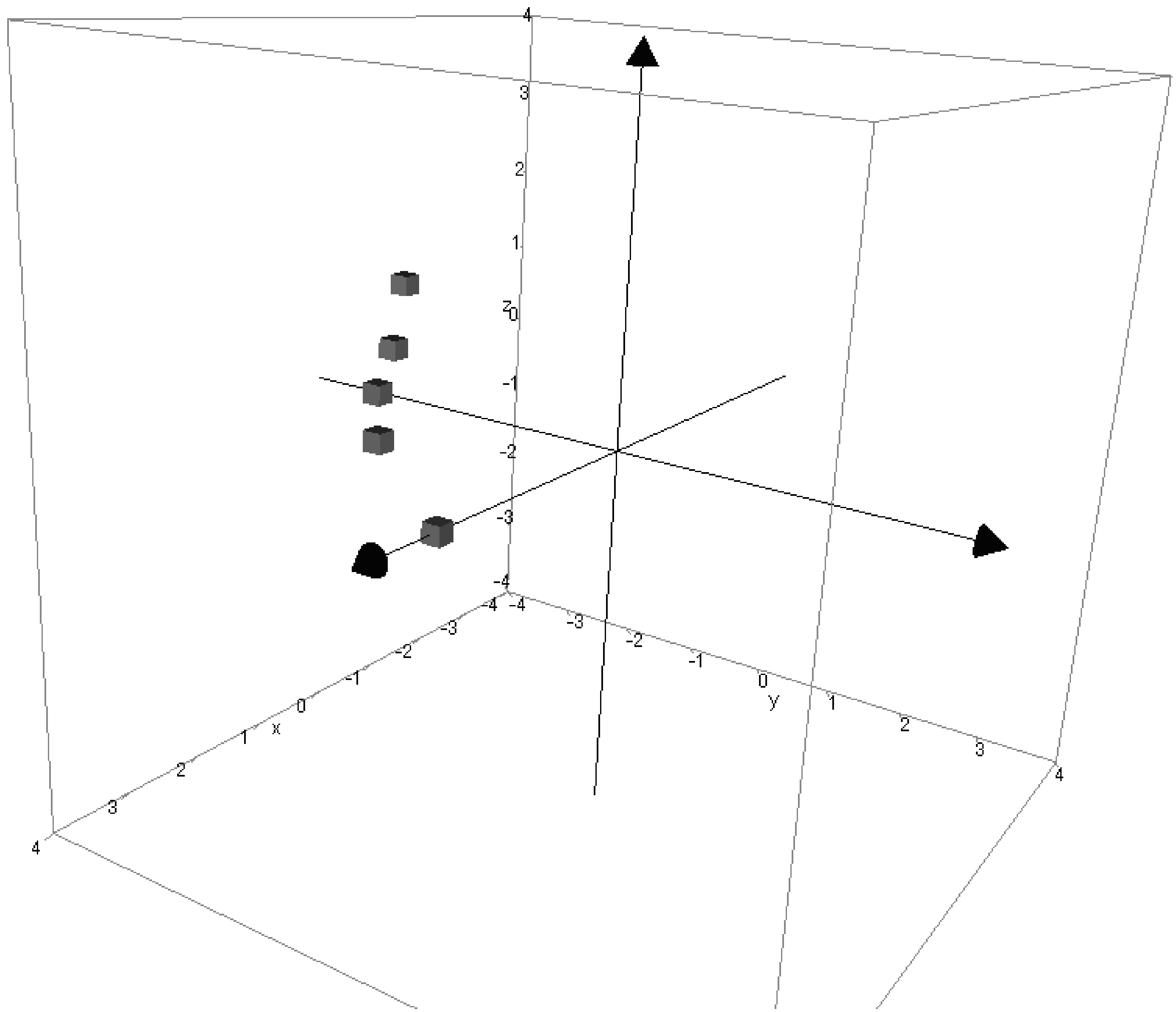
Example: $\vec{r}(t) = (3\cos t)i - (2\sin t)j + tk = \langle 3\cos t, -2\sin t, t\rangle$

is a vetor valued function.

Let us compute a few values of this function

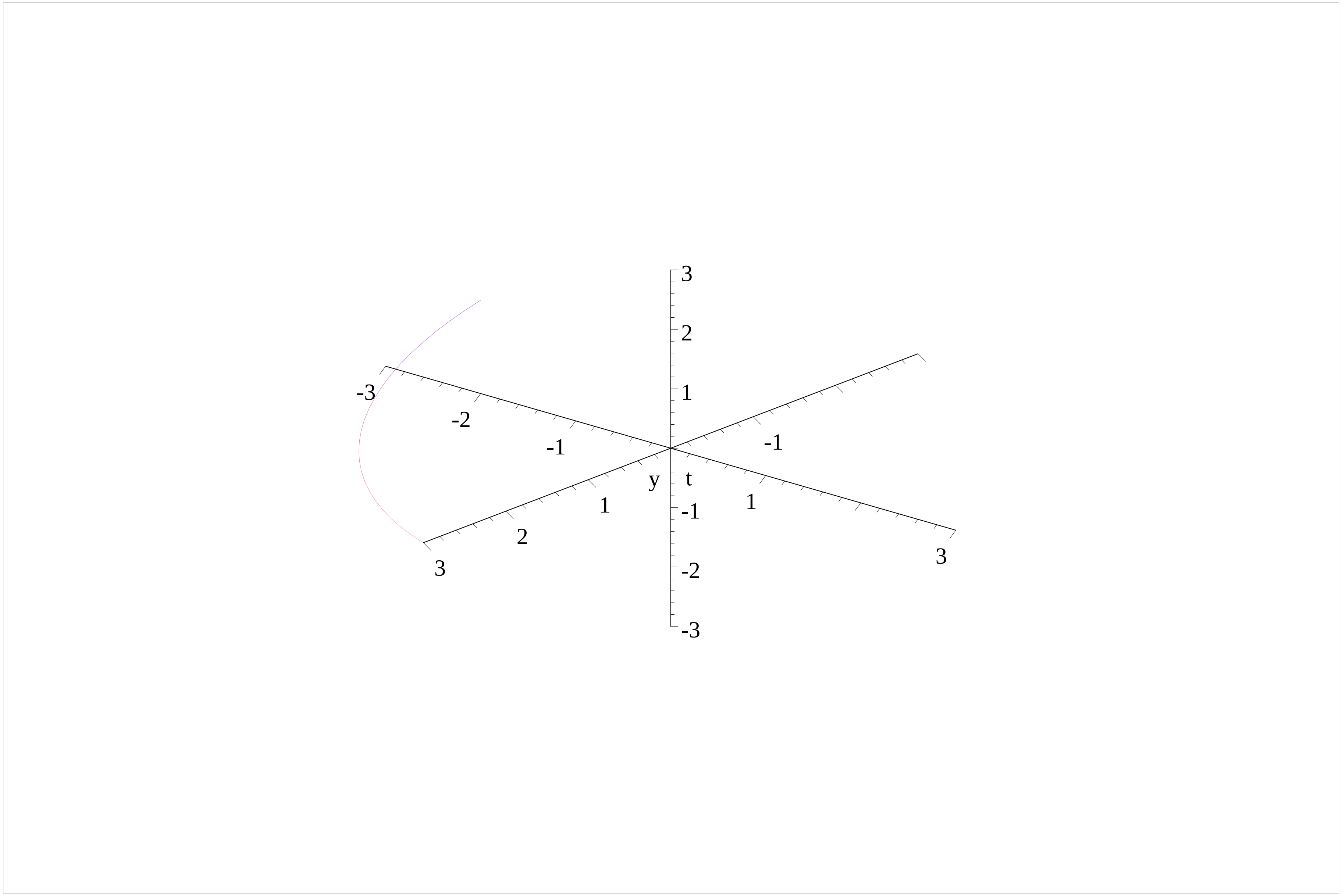
$t = 0$, $\langle 3\cos 0, -2\sin 0, 0\rangle = \langle 3, 0, 0\rangle$
$t = \frac{\pi}{6}$, $\left\langle 3\cos\frac{\pi}{6}, -2\sin\frac{\pi}{6}, \frac{\pi}{6}\right\rangle = \left\langle \frac{3\sqrt{3}}{2}, -1, \frac{\pi}{6}\right\rangle$
$t = \frac{\pi}{4}$, $\left\langle 3\cos\frac{\pi}{4}, -2\sin\frac{\pi}{4}, \frac{\pi}{4}\right\rangle = \left\langle \frac{3}{\sqrt{2}}, -\sqrt{2}, \frac{\pi}{4}\right\rangle$
$t = \frac{\pi}{3}$, $\left\langle 3\cos\frac{\pi}{3}, -2\sin\frac{\pi}{3}, \frac{\pi}{3}\right\rangle = \left\langle \frac{3}{2}, -\sqrt{3}, \frac{\pi}{3}\right\rangle$
$t = \frac{\pi}{2}$, $\left\langle 3\cos\frac{\pi}{2}, -2\sin\frac{\pi}{2}, \frac{\pi}{3}\right\rangle = \left\langle 0, -2, \frac{\pi}{2}\right\rangle$

Look at a plot of these points

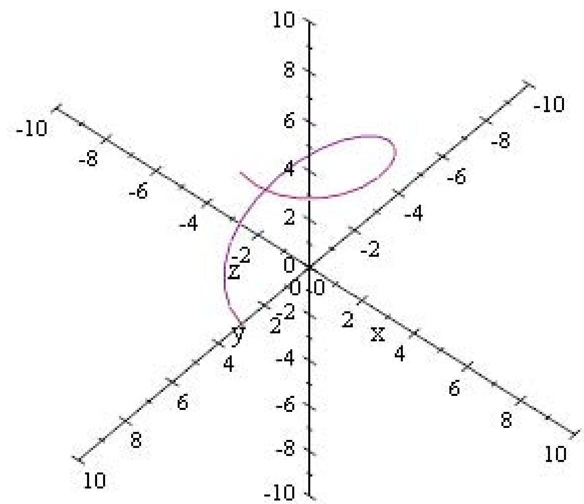


x
y
z

A sketch will look like

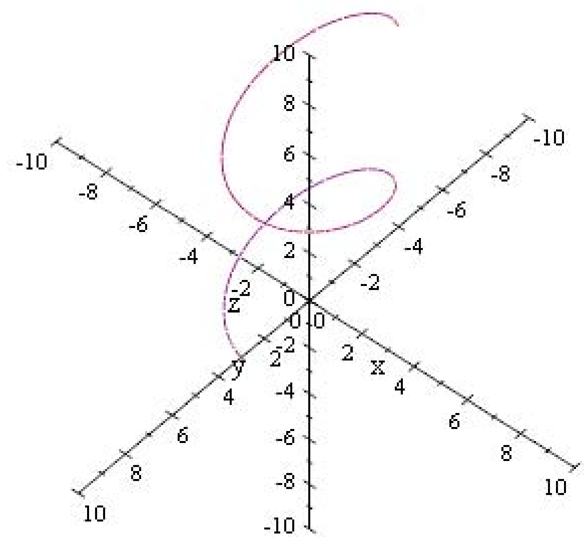

3
2
1
-1
-2
-3
-3
-2
-1
1
2
3
-1
1
3
y
t

If we compute several values for $0 \leq t \leq 2\pi$ and connect them by smooth curve, we shall see



z
x
y

If we compute several values for $0 \le t \le 4\pi$ and connect them by smooth curve, we shall see


z
y
x

If we write the values in terms of the coordinates

$\mathbf{x} = \mathbf{3}\cos\mathbf{t} \qquad \mathbf{y} = -\mathbf{2}\sin\mathbf{t} \qquad \mathbf{z} = \mathbf{t}$

Note that

$x^2 = 9\cos^2 t$ **and** $y^2 = 4\sin^2 t$

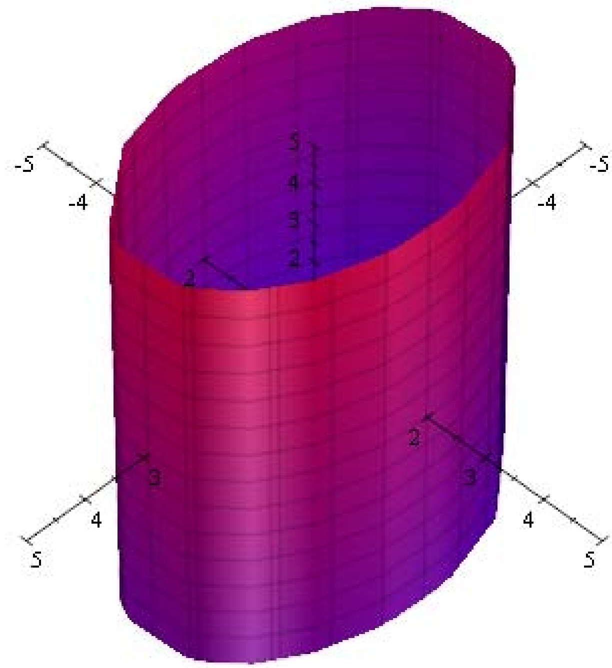
implies

$\frac{x^2}{9} = \cos^2 t$ **and** $\frac{y^2}{4} = \sin^2 t$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

This means that the graph of the curve lies on

the cylinder



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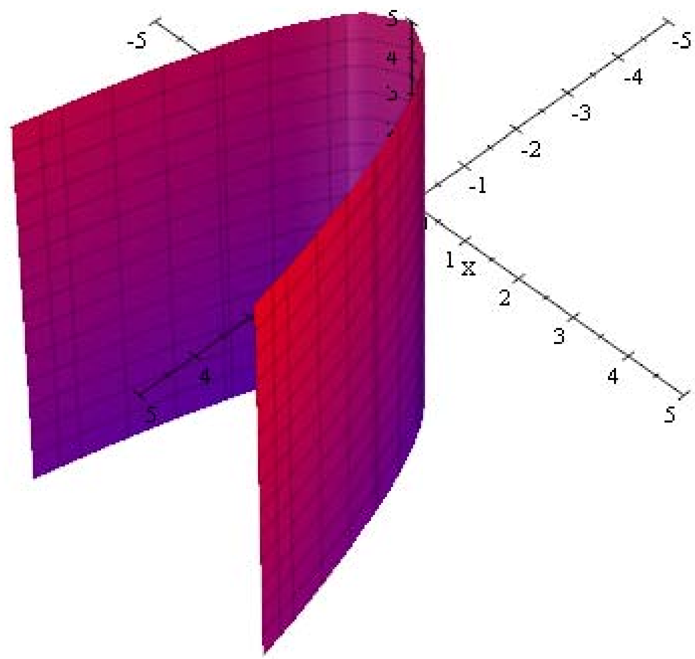
-5
-4
2
3
4
5
3
4
5

As values of t change, the points spiral around this cylinder and the resulting curve is called a helix.

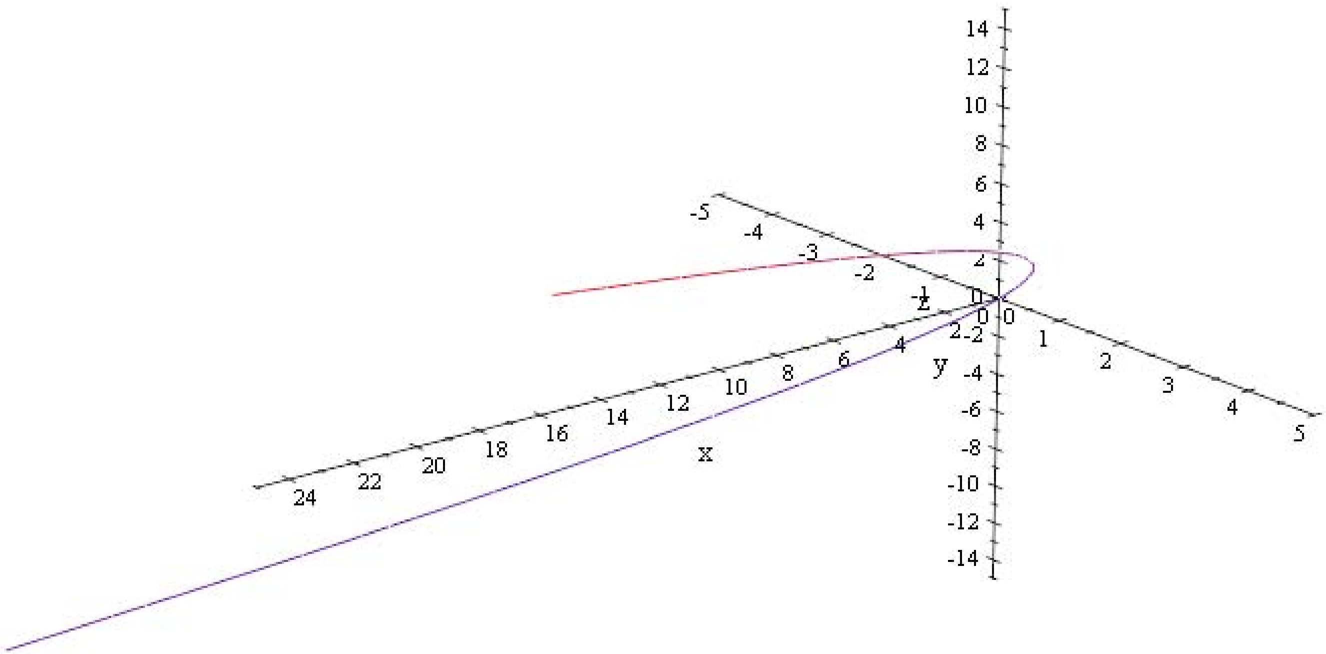
Example 2:

To disuss a graph of $x = t^2, \quad y = t, \quad z = 3t$

The curve lies on the cylinder $x = y^2$



A graph of the curve looks like



Let us look at the excercise #38 on the page 840 of the text book:

$\vec{r}(t)=\langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

$\mathbf{x} = \cos\mathbf{t} + \mathbf{t}\sin\mathbf{t} \rightarrow \mathbf{x}^2 = \cos^2\mathbf{t} + \mathbf{t}^2\sin^2\mathbf{t} + \mathbf{2t}\sin\mathbf{t}\cos\mathbf{t}$
$\mathbf{y} = \sin\mathbf{t} - \mathbf{t}\cos\mathbf{t} \rightarrow \mathbf{y}^2 = \sin^2\mathbf{t} + \mathbf{t}^2\cos^2\mathbf{t} - \mathbf{2t}\sin\mathbf{t}\cos\mathbf{t}$

$\mathbf{x}^2 + \mathbf{y}^2 = \cos^2\mathbf{t} + \sin^2\mathbf{t} + \mathbf{t}^2\sin^2\mathbf{t} + \mathbf{t}^2\cos^2\mathbf{t}$

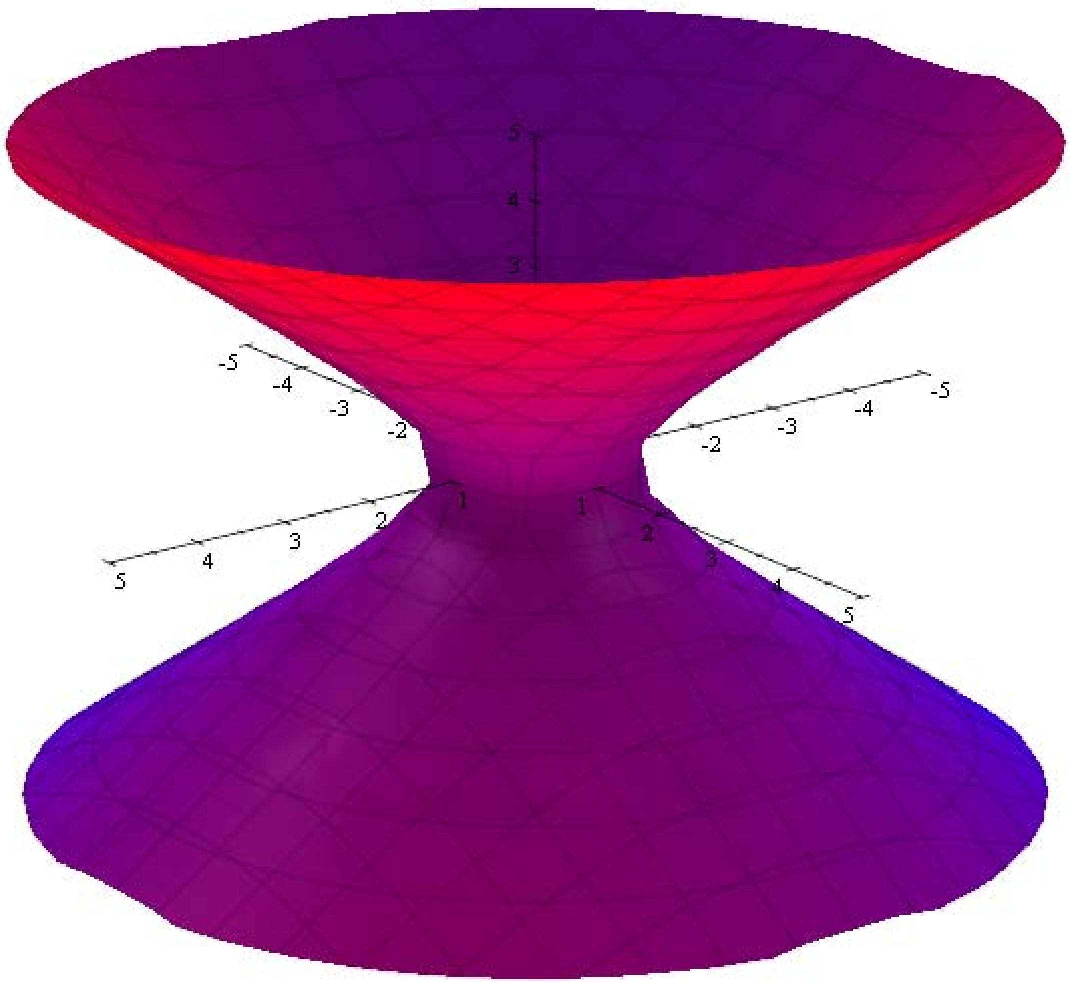
$\mathbf{x}^2 + \mathbf{y}^2 = \mathbf{1} + \mathbf{t}^2$

use $z = t$

$\mathbf{x}^2 + \mathbf{y}^2 = \mathbf{1} + \mathbf{z}^2$

$\mathbf{x}^2 + \mathbf{y}^2 - \mathbf{z}^2 = \mathbf{1}$

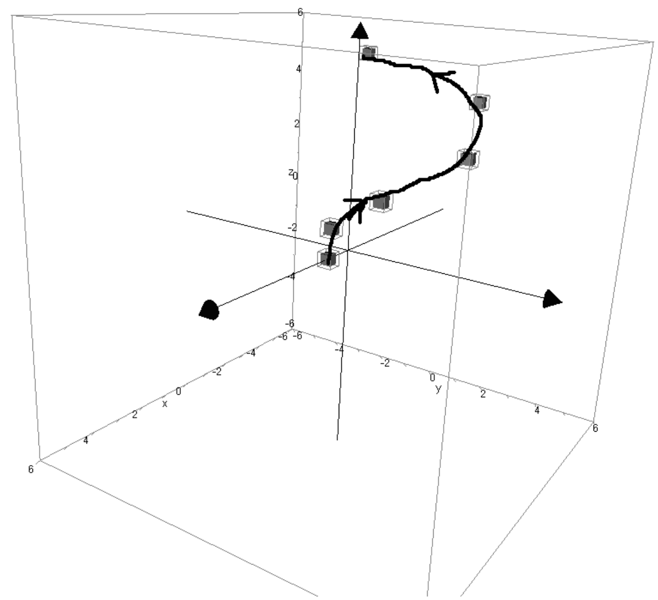
The curves lies on the hyperboloid



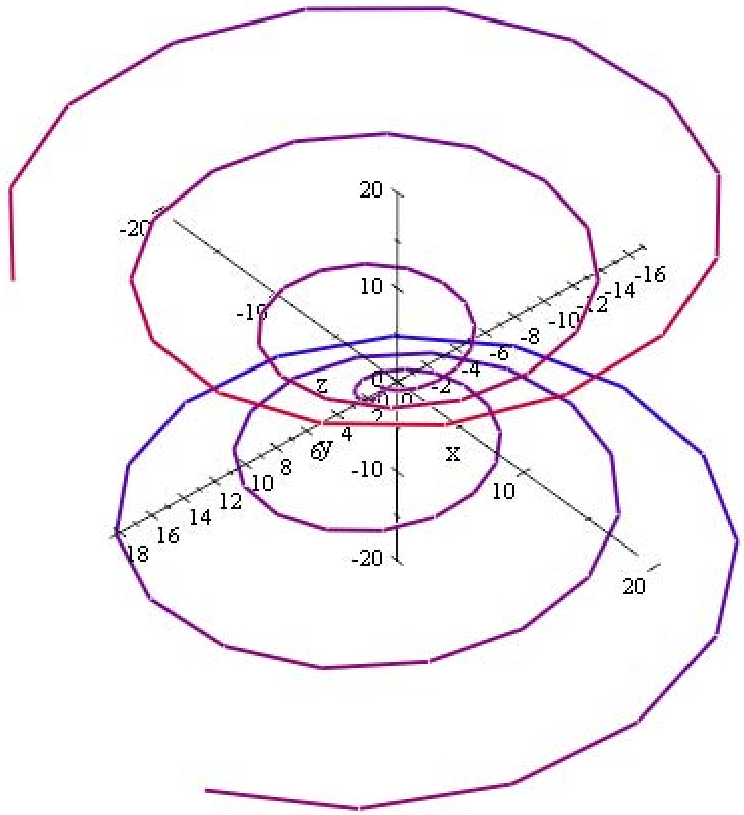
5
4
3
-5
-4
-3
-2
1
2
3
4
5
-5
-4
-3
-2
1
2
3
4
5

t	$\cos t + t\sin t$	$\sin t - t\cos t$	t
0	1	0	0
1	$\cos 1 + \sin 1 = 1.38$	$\sin 1 - \cos 1 = 0.3$	1
2	$\cos 2 + 2\sin 2 = 1.4$	$\sin 2 - 2\cos 2 = 1.74$	2
3	$\cos 3 + 3\sin 3 = -0.57$	$\sin 3 - 3\cos 3 = 3.1$	3
4	$\cos 4 + 4\sin 4 = -3.68$	$\sin 4 - 4\cos 4 = 1.86$	4
5	$\cos 5 + 5\sin 5 = -4.51$	$\sin 5 - 5\cos 5 = -2.38$	5

Based on these points, we have a graph



A graph involving more points will look like



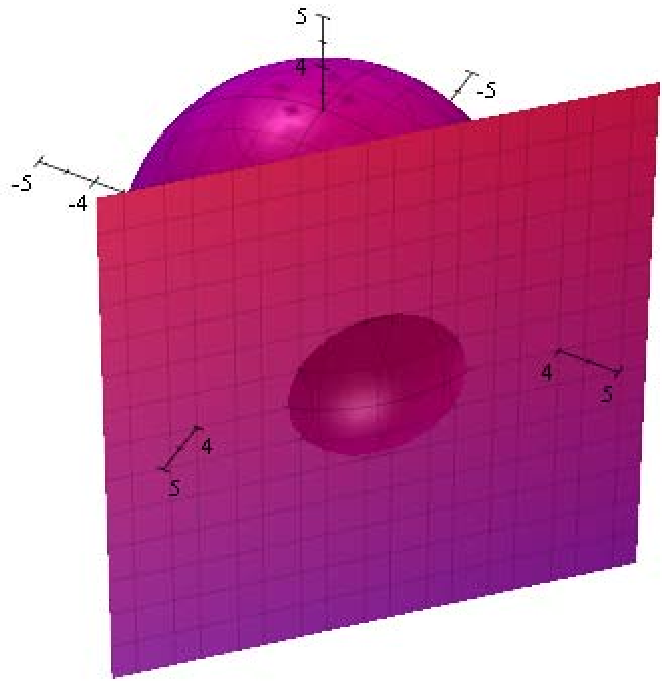
Now, let us turn our attetion to examples involving curves that are obtained by taking intersection of two surfaces.

#64 on the page 838

To write a parametric equation of the space curve that is given by the intersection of

$x^2 + y^2 + z^2 = 10$ **a sphere**
and
$x + y = 4$ **a plane**

We can easily visulalize that if the above two intersect, they intersect along a circle.



We have to write a vector valued function with the parameter given by $x = 2 + \sin t$

Note that we have $x + y = 4$ **means** $2 + \sin t + y = 4 \rightarrow y = 2 - \sin t$

To obtain y**, we can substitute these values in the equation of the sphere** $x^2 + y^2 + z^2 = 10$

$(2 + \sin t)^2 + (2 - \sin t)^2 + \mathbf{z}^2 = \mathbf{10}$

$\rightarrow$

$\mathbf{4} + \mathbf{4}\sin \mathbf{t} + \sin^2 \mathbf{t} + \mathbf{4} - \mathbf{4}\sin \mathbf{t} + \sin^2 \mathbf{t} + \mathbf{z}^2 = \mathbf{10}$

$\rightarrow$

$\mathbf{8} + \mathbf{2}\sin^2 \mathbf{t} + \mathbf{z}^2 = \mathbf{10}$

$\rightarrow$

$\mathbf{z}^2 = \mathbf{10} - \mathbf{8} - \mathbf{2}\sin^2 \mathbf{t}$

$\rightarrow$

$\mathbf{z}^2 = \mathbf{2} - \mathbf{2}\sin^2 \mathbf{t}$

$\rightarrow$

$\mathbf{z}^2 = \mathbf{2}\left(1 - \sin^2 t\right)$

$\rightarrow$

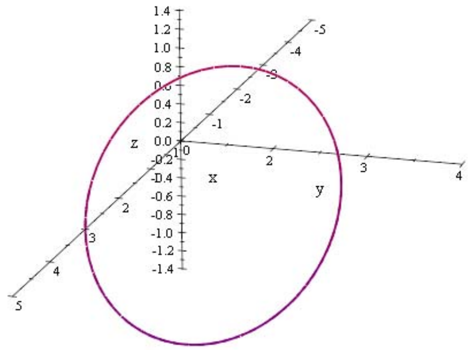
$\mathbf{z}^2 = \mathbf{2}\cos^2 \mathbf{t}$

$\rightarrow$

$\mathbf{z} = \sqrt{2}\cos \mathbf{t}$

$\vec{r}(t) = (2 + \sin t)\mathbf{i} + (2 - \sin t)\mathbf{j} + \sqrt{2}\cos \mathbf{t}\mathbf{k}$

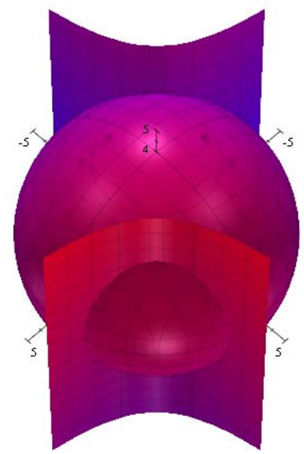
We may write values of the vector function for some values of t



#66 on the page 838

To find the intersection of $x^2 + y^2 + z^2 = 16$ **and** $xy = 4$

the intersection of a sphere and the cylinder given by the hyperbola $xy = 4$
as a generating curve.



-5
5
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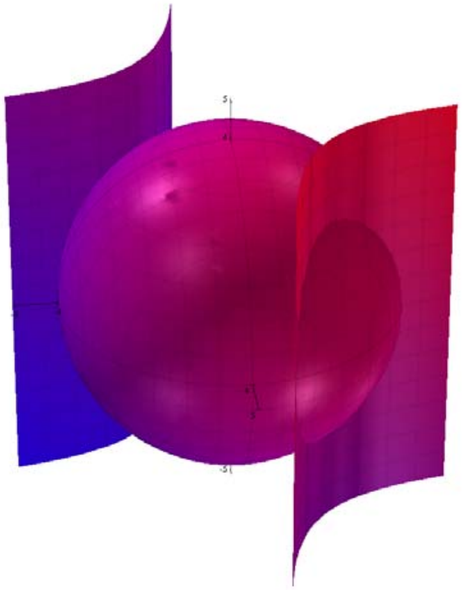
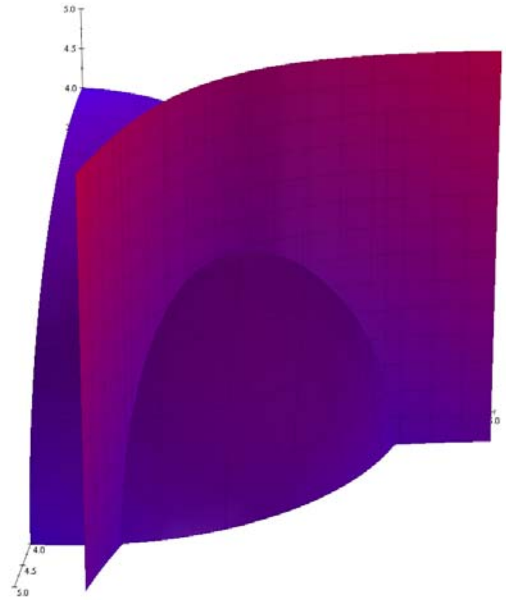


We are confined in the first octant



To write a vector equation with $x = t$

when $x = t$
xy $= 4 \rightarrow$ **y** $= \frac{4}{t}$

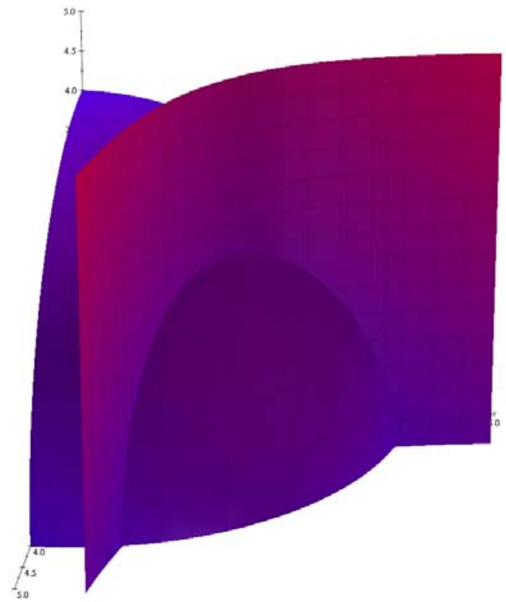
To find a function for z **use**

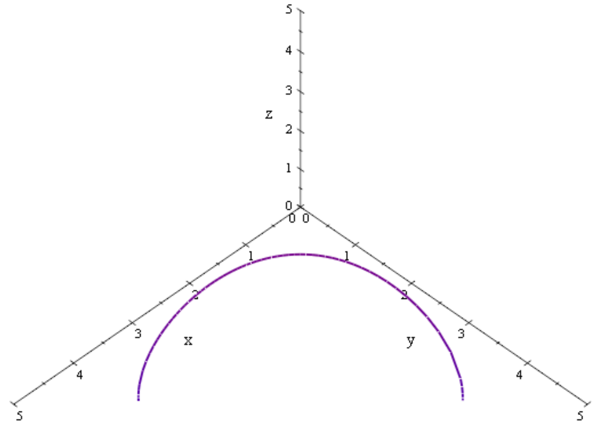
$x^2 + y^2 + z^2 = 16$

$t^2 + \frac{16}{t^2} + z^2 = 16$
$z = \sqrt{16 - t^2 - \frac{16}{t^2}}$

Therefore a vector function is

$\vec{r}(t) = \mathbf{t}\mathbf{i} + \frac{4}{t}\mathbf{j} + \sqrt{16 - t^2 - \frac{16}{t^2}}\,\mathbf{k}$





Remember that the limit and continuity of

$\vec{r}\,(t) = f(t)i + g(t)j + h(t)k$

at a point depends on the limits and continuity of f, g, h
at that point.

Likewise, look at the theorems 12.1 and 12.2 on pages 841 and 842 to connect the derivative of $\vec{r}\ (t)$ with the derivatives of f, g, h and its properties.

Examples:

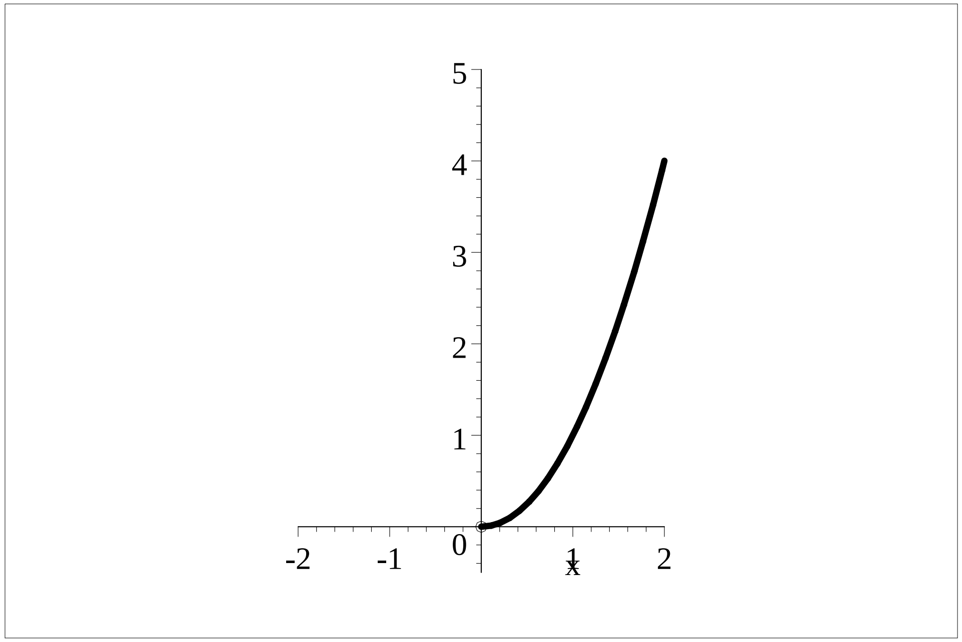
#6 on the page 846

first, we would like to sketch the parametric curve

$\vec{r}(t) = \mathbf{e}^{t}\mathbf{i} + \mathbf{e}^{2t}\mathbf{j}$

the curve lies on the parabola $y = x^2$

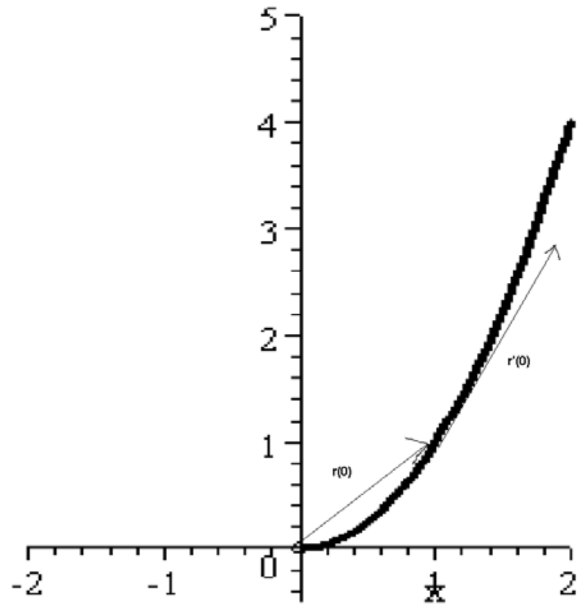
Since $e^t > 0$, we have the graph only in the first quadrant



To find an sketch $\vec{r}(0) = e^0 i + e^{2\times 0} j = i + j = \langle 1, 1\rangle$

$\vec{r}\,'(t) = \mathbf{e}^t\mathbf{i} + \mathbf{2e}^{2t}\mathbf{j}$

$\vec{r}\,'(0) = \mathbf{i} + \mathbf{2j} = \langle 1, 2\rangle$



$\vec{r}'(0)$ **is tangent to the curve at** $(1,1)$**, the point that corresponds to** $t=0$

#16 on the page 846

To find $\vec{r}'(t)$ **if** $\vec{r}(t) = \langle \sin t - t\cos t, \cos t + t\sin t, t^2 \rangle$

$\mathbf{D}_t(\sin t - t\cos t) = \cos \mathbf{t} - \cos \mathbf{t} + \mathbf{t}\sin \mathbf{t} = \mathbf{t}\sin \mathbf{t}$

$\mathbf{D}_t(\cos t + t\sin t) = -\sin \mathbf{t} + \sin \mathbf{t} + \mathbf{t}\cos \mathbf{t} = \mathbf{t}\cos \mathbf{t}$

$\mathbf{D}_t(t^2) =$ **2t**

$\vec{r}'(t) = \langle t\sin t, t\cos t, 2t\rangle$

#26 on the page 846

To find $\vec{r}'(t)$ **,** $\vec{r}''(t)$ **and** $\vec{r}'(t) \cdot \vec{r}''(t)$

$\vec{r}(t)=\langle e^{-t}, t^2, \tan t\rangle$
$\vec{r}'(t)=\langle -e^{-t}, 2t, \sec^2 t\rangle$
$\vec{r}''(t)=\langle e^{-t}, 2, 2\sec t \sec t \tan t\rangle=\langle e^{-t}, 2, 2\sec^2 t \tan t\rangle$

$\vec{r}'(t)\cdot\vec{r}''(t)=\langle -e^{-t}, 2t, \sec^2 t\rangle\cdot\langle e^{-t}, 2, 2\sec^2 t\tan t\rangle= -\mathbf{e}^{-2t}+\mathbf{4t}+\mathbf{2}\sec^4\mathbf{t}\tan\mathbf{t}$

Recall that

$\vec{r}(t)=f(t)i+g(t)j+h(t)k$ **is smooth on an open interval I, if** $f'(t)$**,** $g'(t)$**,** $h'(t)$ **are all continuous on I and** $\vec{r}'(t) \neq 0$ **for any value of** t **in I.**

#38 on the page 846

To find the open intervals on which $\vec{r}(t) = \sqrt{t}\,i + (t^2-1)j + \frac{1}{4}tk$ **is smooth**

$\vec{r}'(t)=\frac{1}{2\sqrt{t}}\mathbf{i}+\mathbf{2tj}+\frac{1}{4}\mathbf{k}$

Function given by $\frac{1}{2\sqrt{t}}$ **is continuous on** $(0,\infty)$

Function given by $2t$ **is continuous on** $(-\infty,\infty)$

Function given by $\frac{1}{4}$ is continuous on $(-\infty, \infty)$

Note that $\vec{r}\,'(t) \neq 0$ **on** $(0, \infty)$

Therefore $\vec{r}$ **is smooth on** $(0, \infty)$

#54 on the page 847

$\int (e^t i + \sin t j + \cos t k)$**dt**
$=\left(\int e^t dt\right) i + \left(\int \sin t dt\right) j + \left(\int \cos t dt\right) k$
$=e^t i - \cos t j + \sin t k + C$

Suggested Practice:

Section 12.1: 5,7,11,13,15,17,19,25,31,35,37,49,61,65,75,77
Section 12.2: 7,9,13,15,17,23,25,31,37,39,41,43,55,59,65,67