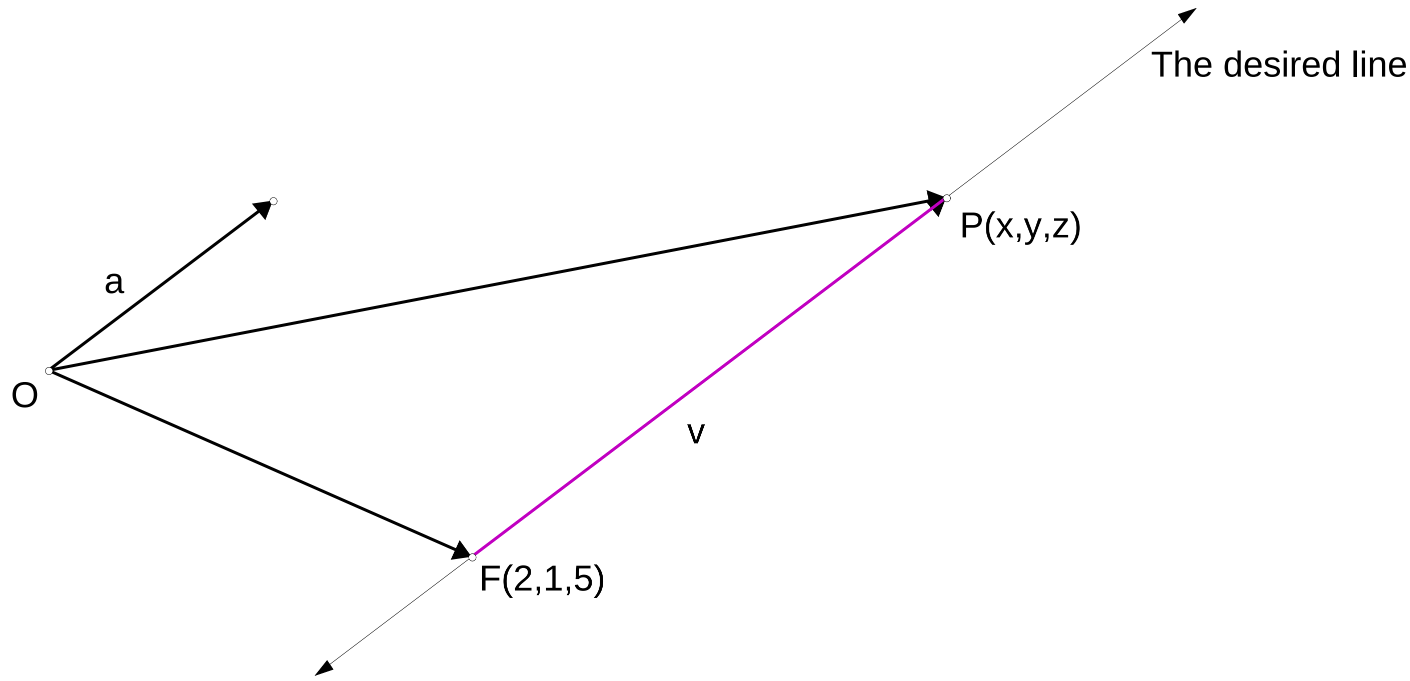
Lesson 2 Part 3

To write an equation of the line that passes through the point F with coordinates (2,1,5) and is parallel to the vector $a = \langle a_1, a_2, a_3 \rangle$



We can think of the desired line as the set of points P(x,y,z) such that the vector v=$\overrightarrow{FP}$ **is parallel to the vector** $a = \langle a_1, a_2, a_3 \rangle$

$a = \langle a_1, a_2, a_3 \rangle$ is called the direction vector of the line and a_1, a_2, a_3 a set of direction numbers

$\overrightarrow{FP} = \langle x-2, y-1, z-5\rangle$

Since this vector is parallel to $a = \langle a_1, a_2, a_3\rangle$

we should be able to find a scalar t **such that**

$\langle x-2, y-1, z-5\rangle = \mathbf{t}\langle a_1, a_2, a_3\rangle$

OR

$\mathbf{x - 2 = ta_1}$ $\quad$ $\mathbf{y - 1 = ta_2}$ $\quad$ $\mathbf{z - 5 = ta_3}$

that is

$x = 2 + ta_1$ $\quad$ $y = 1 + ta_2$ $\quad$ $z = 5 + ta_3$ **is a parametric form of the equation**

Example 1:

Find an equation of the line that passes through the point $(3, -7, 2)$ **and is parallel to the vector** $\langle 2, -1, 5\rangle$

As shown above, for a typical point with the coordinates (x, y, z) **, we have**

$\langle x-3, y+7, z-2\rangle = \mathbf{t}\langle 2, -1, 5\rangle$

$\mathbf{x - 3 = 2t}$ $\quad$ $\mathbf{y + 7 = -t}$ $\quad$ $\mathbf{z - 2 = 5t}$

is a set of parametric equations

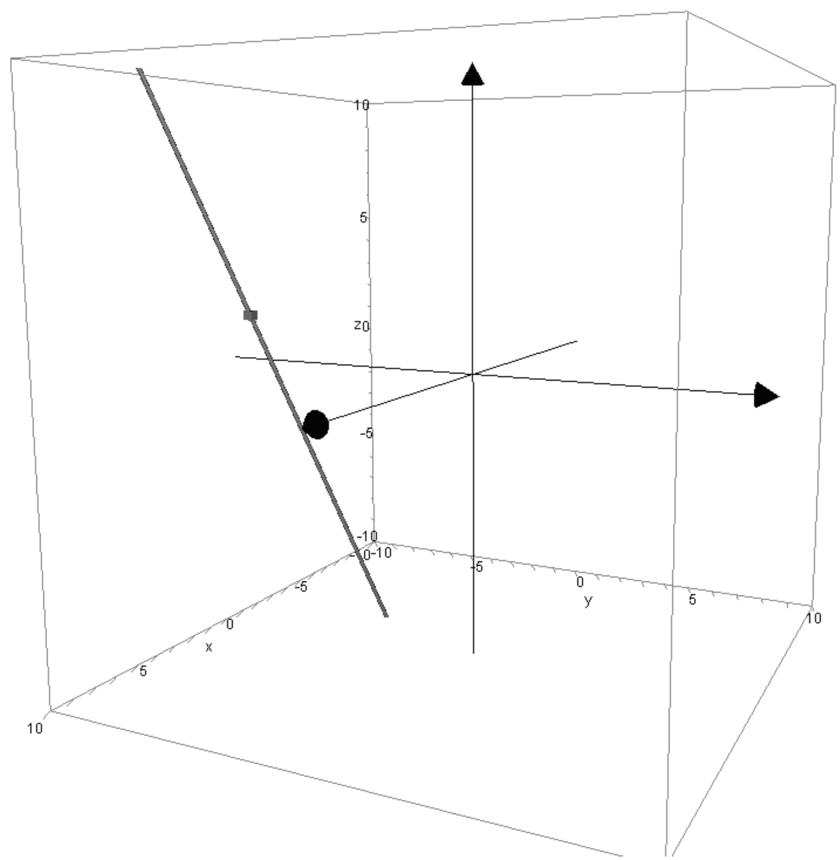
Note that none of the direction numbers $2, -1, 5$ **is 0**

therefore

$$x - 3 = 2t \qquad y + 7 = -t \qquad z - 2 = 5t$$

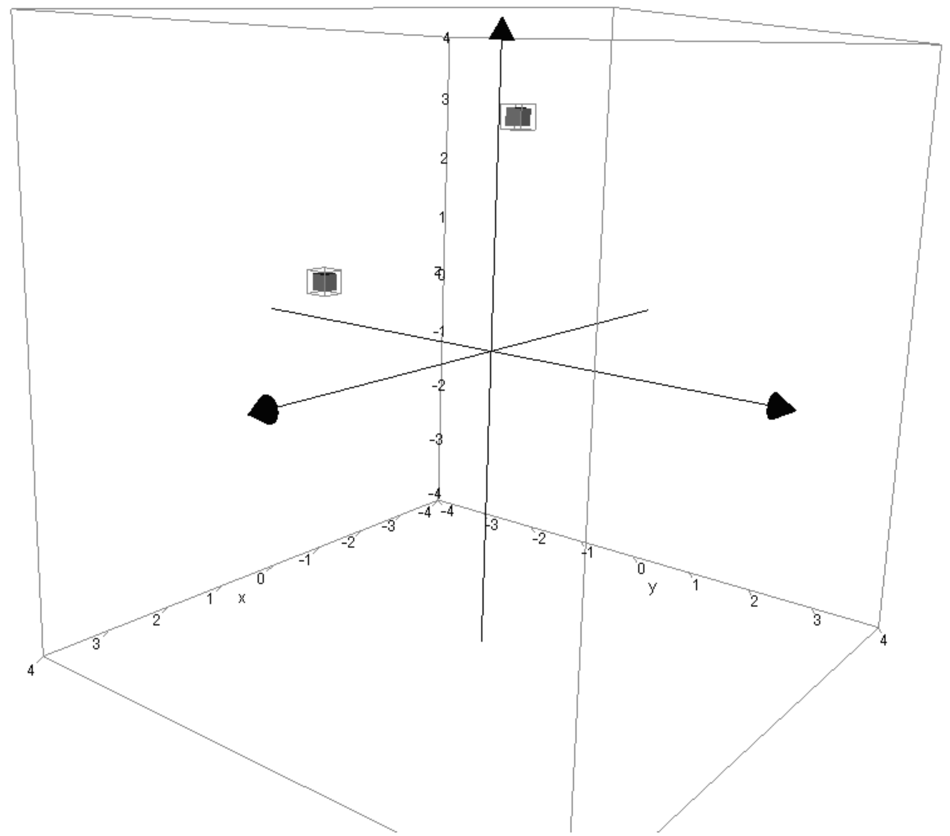
may be rewritten as

$\boxed{\frac{x-3}{2} = \frac{y+7}{-1} = \frac{z-2}{5} \quad \text{called symmetric equations of the same line}}$

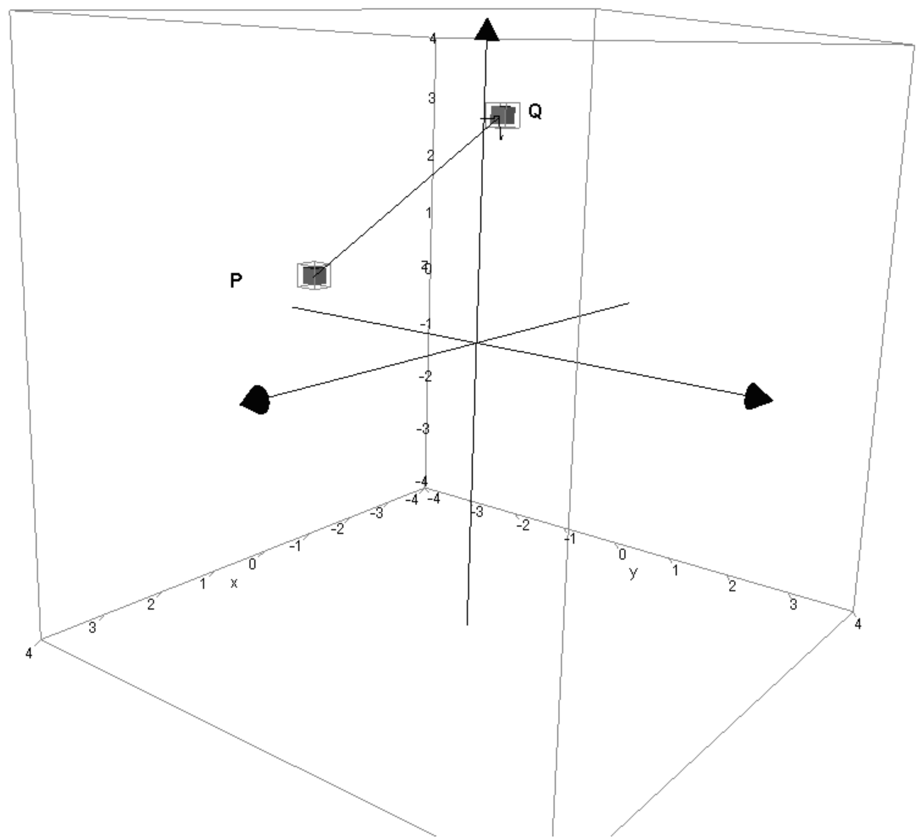


Example 2:

To write equation of a line that passes through the points $P(2,-1,1)$ **and** $Q(1,1,3)$



$\overrightarrow{PQ}$ **may be taken as a direction vector**

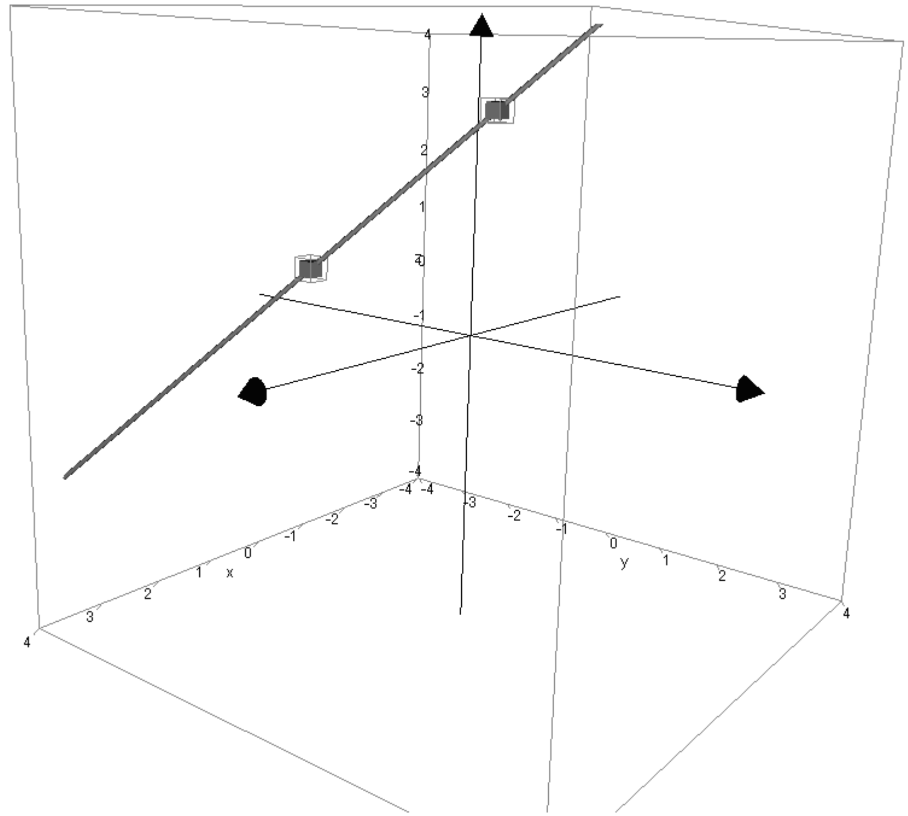


$\overrightarrow{PQ}=\langle 1-2, 1-(-1), 3-1\rangle=\langle -1,2,2\rangle$

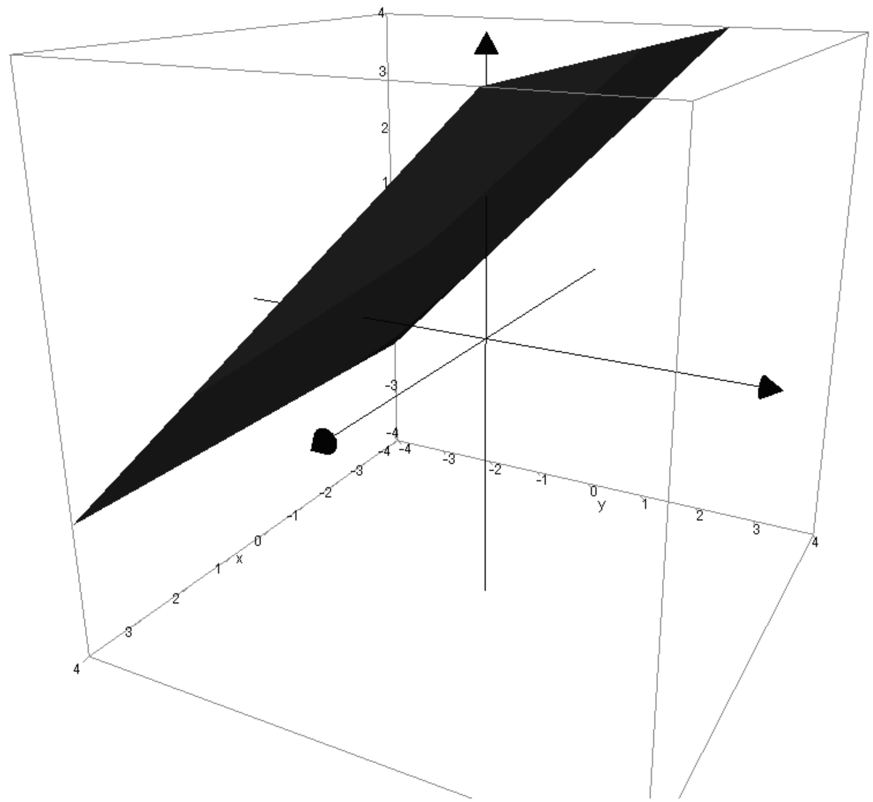
A line passing through $(2,-1,1)$ **and parallel to** $\langle -1,2,2\rangle$

has parametric equations

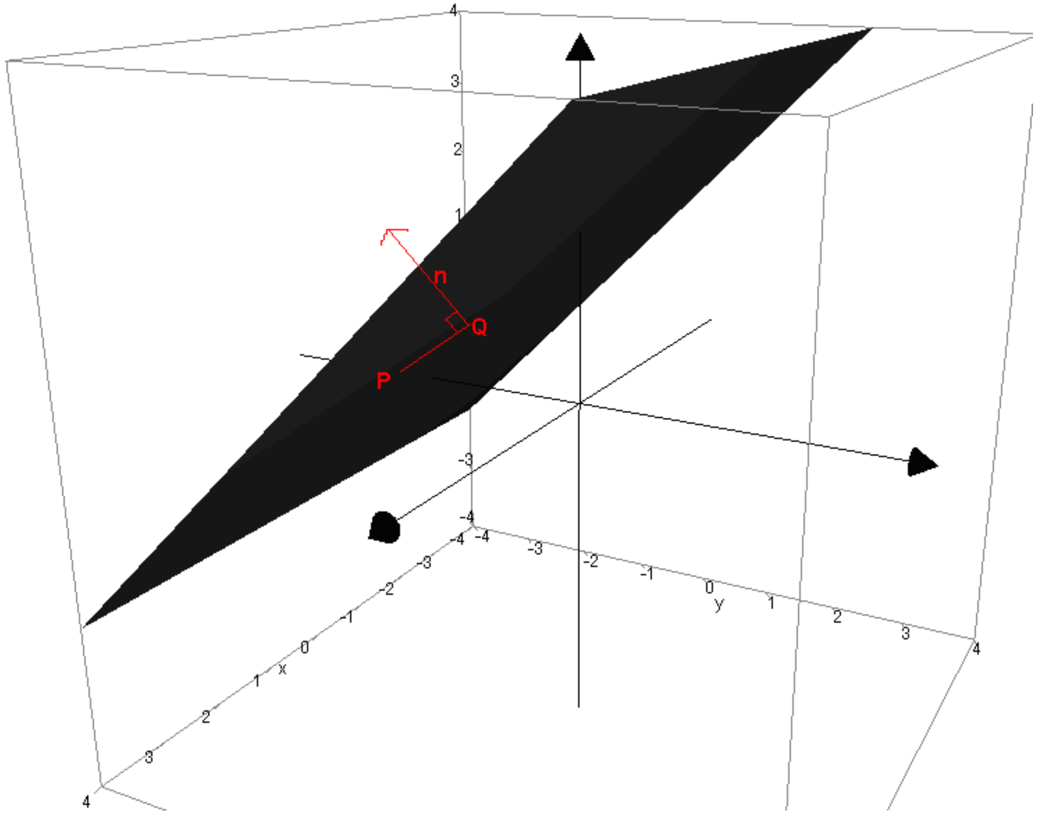
$\mathbf{x} = \mathbf{2} - \mathbf{t}$ $\quad\quad$ $\mathbf{y} = -\mathbf{1} + \mathbf{2t}$ $\quad\quad$ $\mathbf{z} = \mathbf{1} + \mathbf{2t}$

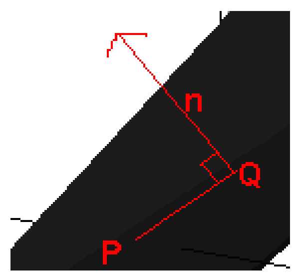


Equation of a plane



Assume that we are given that the plane passes through a point Q(x_o,y_o,z_0) and is perpendicular to the vector n $=\langle a,b,c\rangle$


n
Q
P
x
y


n
Q
P

If we take a point P(x, y, z) **on this plane**

then $\overrightarrow{PQ} \perp n$

or

$\overrightarrow{PQ} \cdot \mathbf{n} = \mathbf{0}$

That is

$\langle x - x_0, y - y_o, z - z_o \rangle \cdot \langle a, b, c \rangle = \mathbf{0}$

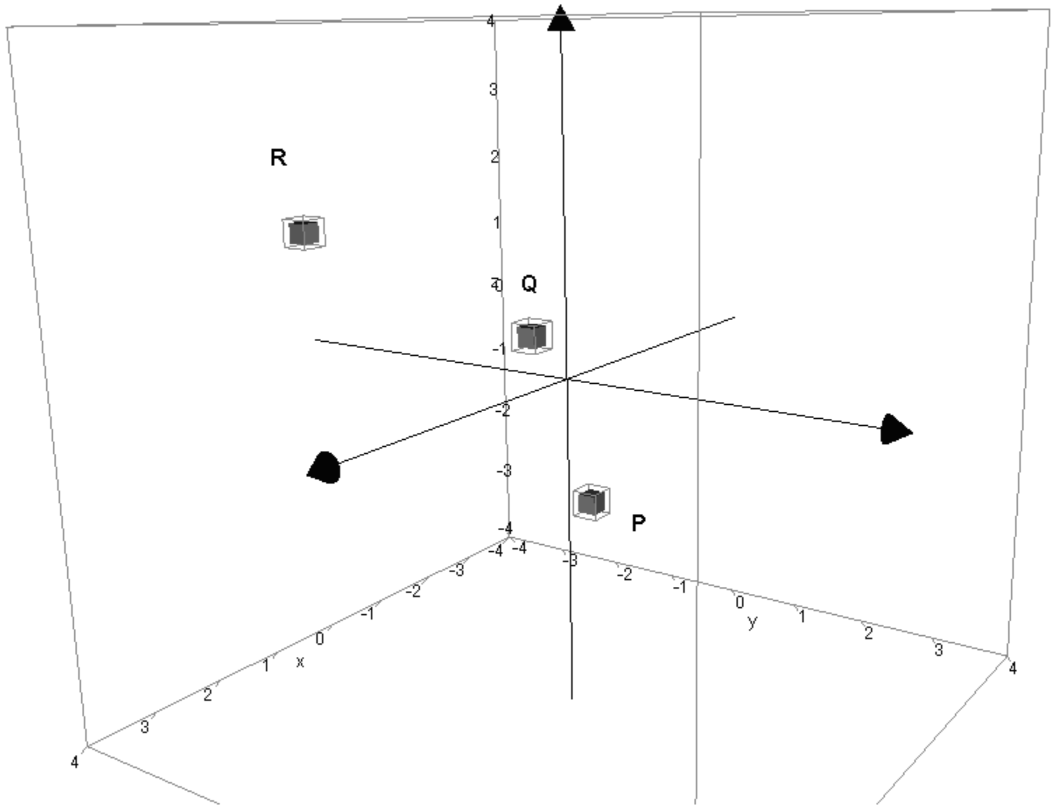
$\boxed{a(x - x_0) + b(y - y_o) + c(z - z_o) = 0 \quad \text{STANDARD EQUATION OF A PLANE}}$

This equation may be transformed in the form

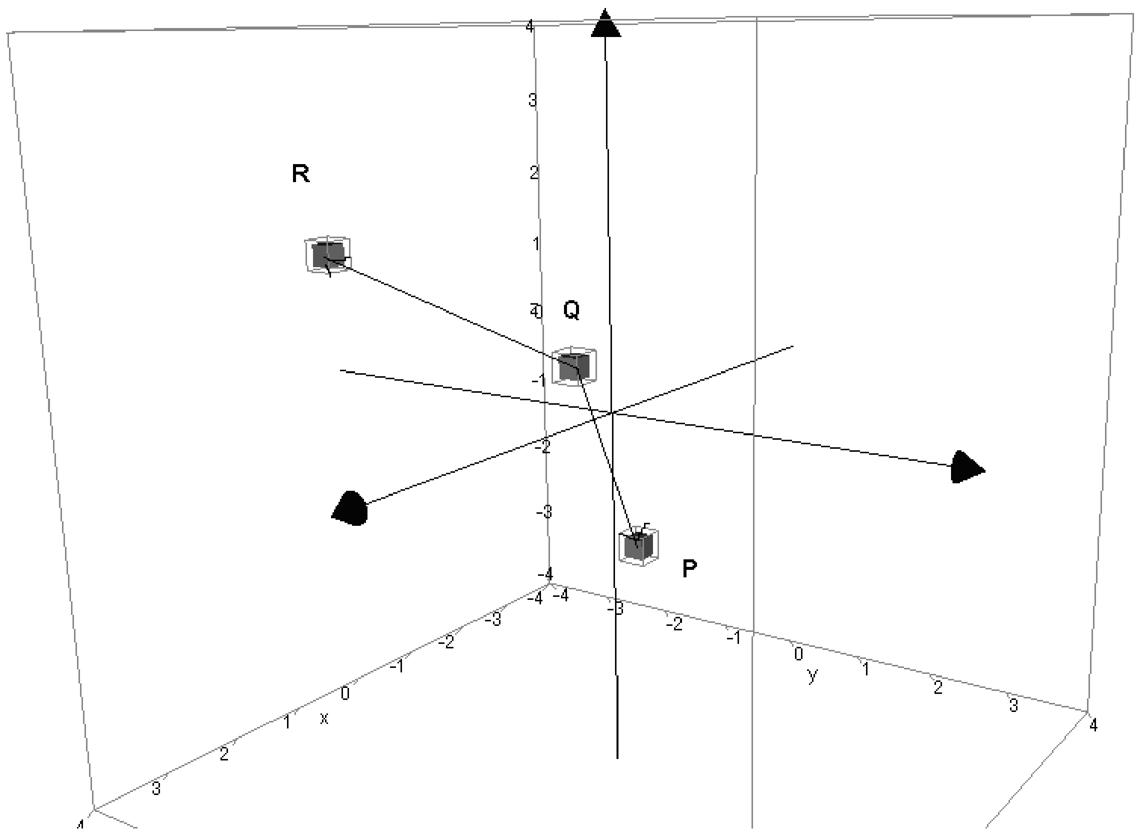
$ax + by + cz + d = 0$ **General form of Equation of a plane**

Example 3:

Find an equation of the plane that contains the points P$(1, 1, -1)$, **Q**$(2, 1, 1)$ **and R**$(3, -1, 2)$



We can use the above procedure if we can find *n* that is a vector perpendicular to the plane



Note that $\overrightarrow{QP} \times \overrightarrow{QR}$ is perpendicular to both the vectors and consequently perpendicular to the plane containing P,Q, and R

Recall
P$(1,1,-1)$,**Q**$(2,1,1)$ **and R**$(3,-1,2)$

$\overrightarrow{QP}=\langle 2-1, 1-1, 1-(-1)\rangle=\langle 1,0,2\rangle$

$\overrightarrow{QR} = \langle 3-2, -1-1, 2-1 \rangle = \langle 1, -2, 1 \rangle$

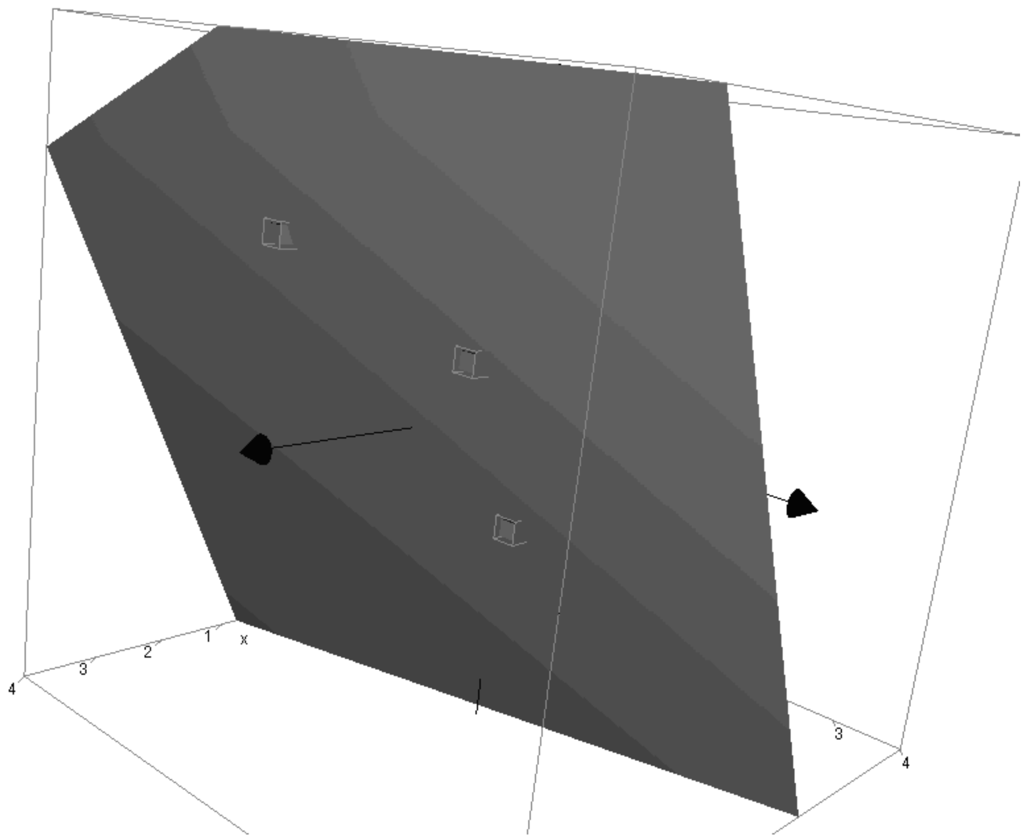
$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 1 & -2 & 1 \end{vmatrix} = (0+4)i + (2-1)j + (-2-0)k = 4i + j - 2k$$

$\overrightarrow{QP} \times \overrightarrow{QR} = \langle 4, 1, -2 \rangle$

Using any of the three points P, Q, R will yield the same plane

$\mathbf{4}(x-1) + \mathbf{1}(y-1) - \mathbf{2}(z-(-1)) = \mathbf{0}$
$4(x-1) + 1(y-1) - 2(z+1) = 0$
$\mathbf{4x - 4 + y - 1 - 2z - 2 = 0}$

$\mathbf{4x + y - 2z - 7 = 0}$



Example 4:

The angle θ between the planes

$\mathbf{x - y + 2z + 3 = 0}$

and

2x $+$ **y** $-$ **z** $+$ **1** $=$ **0**

may be determined by determining the angle between the vectors normal to these planes.

We shall take θ **such that** $0 \leq \theta \leq \frac{\pi}{2}$

The vectors normal to the above two planes are

$\mathbf{n}_1 = \langle 1, -1, 2 \rangle$

$\mathbf{n}_2 = \langle 2, 1, -1 \rangle$

respectively

Remember that

$\mathbf{n}_1 \cdot \mathbf{n}_2 = \|n_1\| \|n_2\| \cos\theta$

or

$$\cos\theta = \frac{n_1 \cdot n_2}{\|n_1\| \|n_2\|}$$

Since we would take $0 \leq \theta \leq \frac{\pi}{2}$

$$\cos\theta = \frac{|n_1 \cdot n_2|}{\|n_1\| \|n_2\|}$$

or

$\cos\theta = \frac{|\langle 1,-1,2\rangle \cdot \langle 2,1,-1\rangle|}{\sqrt{1^2+(-1)^2+2^2}\sqrt{2^2+1^2+(-1)^2}}$

$\cos\theta = \frac{1}{6}$

$\theta = \cos^{-1}\left(\frac{1}{6}\right)$

$\theta \cong 1.4033482475752072887$ **radians**

or $1.4033482475752072887 \times \frac{180}{\pi} = 80.405931773139538556^\circ$

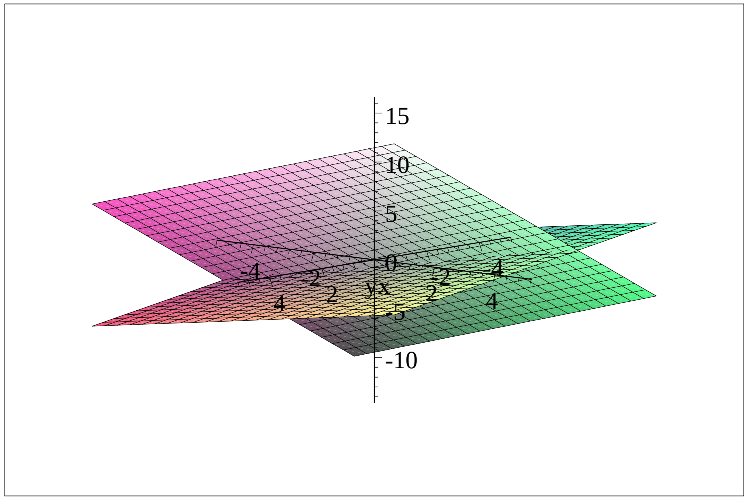
Example 5:

Find an equation of the line of intersection of

$\mathbf{x - y + 2z + 3 = 0}$

and

$\mathbf{2x + y - z + 1 = 0}$



One way to find an equation of the line of intersection is to eliminate one of the three variables between the two equations first

$\mathbf{x - y + 2z + 3 = 0}$

$2x + y - z + 1 = 0$ **adding the two equations will eliminate** y

Add

$\mathbf{3x + z + 4 = 0}$

or

$\mathbf{x} = -\frac{z+4}{3}$

or

$\mathbf{x} = \frac{z+4}{-3}$

Substitute this value in the equation of any of the two planes,

take $x - y + 2z + 3 = 0$

$\frac{z+4}{-3} - \mathbf{y} + \mathbf{2z} + \mathbf{3} = \mathbf{0}$

$z + 4 + 3y - 6z - 9 = 0$

$\mathbf{3y - 5z - 5 = 0}$

$\mathbf{3y = 5z + 5}$

$\mathbf{3y = 5}(z+1)$

$\mathbf{y} = \frac{5(z+1)}{3}$

recall that we already have

$\mathbf{x} = \frac{z+4}{-3}$

Therefore along the line of intersection

x $= \frac{z+4}{-3}$
y $= \frac{5(z+1)}{3}$

Now, we can find coordinates of two points on the line by assigning any two values to z

Take $z = 2$ $x = \frac{2+4}{-3} = -2$ $y = \frac{5(2+1)}{3} = 5$ $(-2, 5, 2)$

Take $z = 5$ $x = \frac{5+4}{-3} = -3$ $y = \frac{5(5+1)}{3} = 10$ $(-3, 10, 5)$

Just to verfiy our calculations, let us verify if these points are on the intersection of

$x - y + 2z + 3 = 0$
$2x + y - z + 1 = 0$

$(-2, 5, 2)$
$-2 - 5 + 2(2) + 3 = 0$
$2(-2) + 5 - 2 + 1 = 0$

GOOD

$(-3, 10, 5)$
$-3 - 10 + 2(5) + 3 = 0$
$2(-3) + 10 - 5 + 1 = 0$
GOOD

To write equation of the line through $(-2, 5, 2)$ **and** $(-3, 10, 5)$

a vector parallel to this line is $\langle -3-(-2), 10-5, 5-2 \rangle = \langle -1, 5, 3 \rangle$

Parametric equation is

$x = -2 - t$ $\quad\quad$ $y = 5 + 5t$ $\quad\quad$ $z = 2 + 3t$

Read the Theorems 11.13 and 11.14

Worked out exercises from the Text:

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To find a parametric equation of the line that passes through the point $(-4, 5, 2)$ **and is perpendicular to the plane given by**

$-x + 2y + z = 5$

A vector perpendicular to $-x + 2y + z = 5$ **is** $\langle -1, 2, 1 \rangle$

therefore the line is parallel to $\langle -1, 2, 1 \rangle$

Since the line passes through $(-4, 5, 2)$

a parametric equation is

$x = -4 - t$ $\quad\quad$ $y = 5 + 2t$ $\quad\quad$ $z = 2 + t$

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To find a parametric equation of the line that passes through the point $(-6, 0, 8)$

and is parallel to the line $x = 5 - 2t \quad y = -4 + 2t \quad z = 0$

Note that $\langle -2, 2, 0 \rangle$ **is a vector parallel to** $x = 5 - 2t \quad y = -4 + 2t \quad z = 0$

Therefore $\langle -2, 2, 0 \rangle$ **is a vector parallel to the desired line**

The line passes through $(-6, 0, 8)$

therefore a parametric equation is $x = -6 - 2t \quad y = 2t \quad z = 8$

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First, we have to determine whether the lines

$\mathbf{x} = -\mathbf{3t} + \mathbf{1} \qquad \mathbf{y} = \mathbf{4t} + \mathbf{1} \qquad \mathbf{z} = \mathbf{2t} + \mathbf{4}$

$\mathbf{x} = \mathbf{3s} + \mathbf{1} \qquad \mathbf{y} = \mathbf{2s} + \mathbf{4} \qquad \mathbf{z} = -\mathbf{s} + \mathbf{1}$

If they intersect, we have to find the angle between the two lines

Intersection quesion:

If the two lines intersect, we should be able to find values of s **and** t **for which**

$-3t + 1 = 3s + 1$ **...... (1)**
$4t + 1 = 2s + 4$ **........ (2)**
$2t + 4 = -s + 1$ **....... (3)**

Add 2 times equation (3) from the equation (2) to get

8t $+$ **9** $=$ **6** $\to$ **t** $= -\frac{3}{8}$

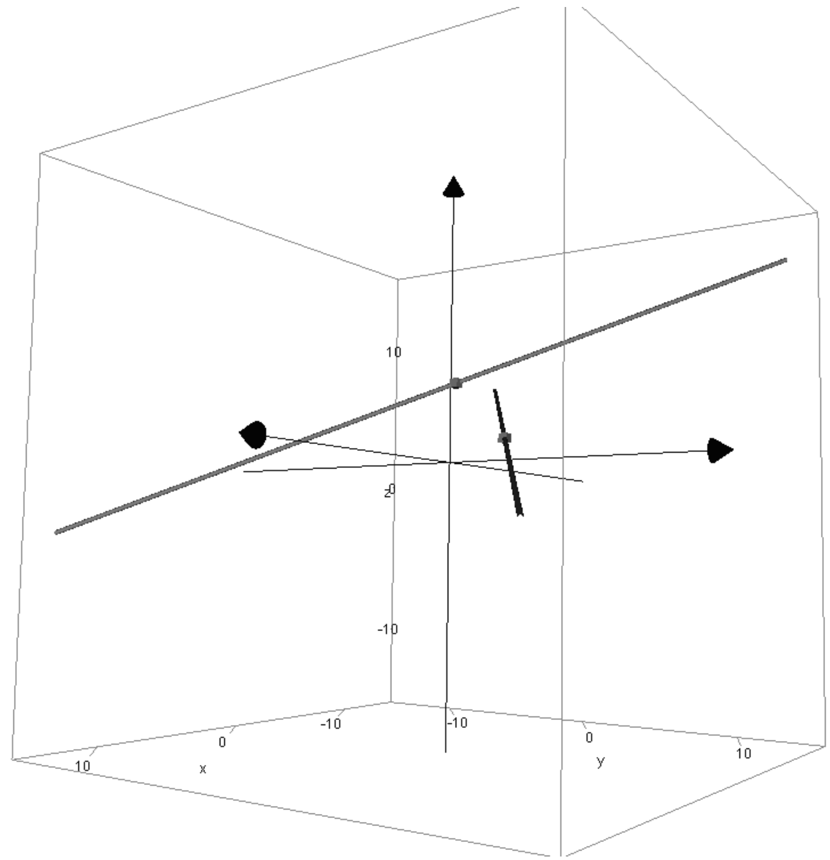
substitute in the equation (2)

$4\left(-\frac{3}{8}\right) + 1 = 2s + 4$**, Solution is:** $\left\{s = -\frac{9}{4}\right\}$

substitute $t = -\frac{3}{8}$ **and** $s = -\frac{9}{4}$ **in equation (1)**

$-3\left(-\frac{3}{8}\right) + 1 = 3\left(-\frac{9}{4}\right) + 1$ **is false**

That means that the two lines do not intersect



Please work on the practice problems

13,19,27,29,39,45,51,59,75,81,85,89,93 in the section 11.5