Lesson 2 Part2 for Section 11.4

I am writing a brief introduction of the determinants for those of you who have not worked with the determinants before.

A 2x2 determinant is $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ **and is evaluated by using the equation**

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \mathbf{ad} - \mathbf{bc}$$

Example 1:

Evaluate

$$\begin{vmatrix} 2 & 1 \\ 3 & -5 \end{vmatrix} = \mathbf{2} \times (-5) - \mathbf{1} \times \mathbf{3} = \mathbf{-13}$$

A 3×3 **determinant can be written in terms of** 2×2 **determinants in the following manner**

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \mathbf{a}_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{a}_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{a}_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= \mathbf{a}_1(b_2c_3 - b_3c_2) - \mathbf{a}_2(b_1c_3 - b_3c_1) + \mathbf{a}_3(b_1c_2 - b_2c_1)$$

Example 2:

Evaluate $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & -6 \\ 1 & 1 & 5 \end{vmatrix}$

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & -6 \\ 1 & 1 & 5 \end{vmatrix}$$
$$= \mathbf{2}(1 \times 5 - (-6) \times 1) - (-1)(3 \times 5 - (-6) \times 1) + \mathbf{3}(3 \times 1 - 1 \times 1)$$
$$= \mathbf{2}(5 + 6) + (15 + 6) + \mathbf{3}(3 - 1)$$
$$= \mathbf{2}(11) + \mathbf{21} + \mathbf{6}$$
$$= \mathbf{22} + \mathbf{21} + \mathbf{6}$$
$$= \mathbf{49}$$

Remember that $i = \langle 1, 0, 0 \rangle$ $\quad j = \langle 0, 1, 0 \rangle$ $\quad k = \langle 0, 0, 1 \rangle$

For three dimensional vectors

$u =< u_1, u_2, u_3 >$ **and** $v =< v_1, v_2, v_3 >$

The cross product $u \times v$ **is defined as**

$$\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

Or we may write in the component form:

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

Example 1:

To find the cross product of the vectors

$\mathbf{u} = \mathbf{2i} - \mathbf{3j} + \mathbf{k}$

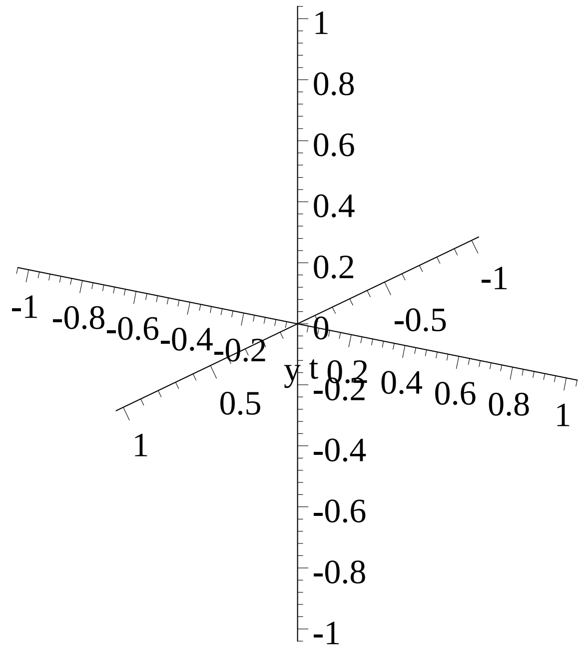
and

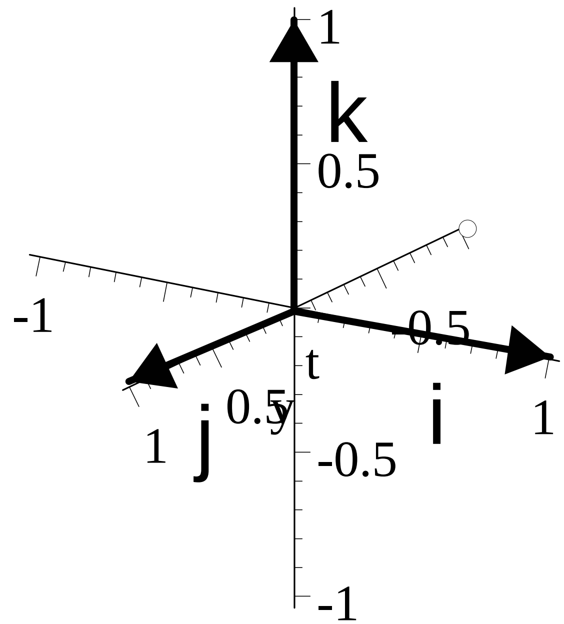
$\mathbf{v} = \mathbf{i} - \mathbf{2j} + \mathbf{7k}$

$$\mathbf{u} \times \mathbf{v}$$
$$= \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 1 & -2 & 7 \end{vmatrix}$$
$$= ((-3) \times 7 - 1 \times (-2))\mathbf{i} + (1 \times 1 - 2 \times 7)\mathbf{j} + (2 \times (-2) - 1 \times (-3))\mathbf{k}$$
$$= -\mathbf{19i} - \mathbf{13j} - \mathbf{k}$$

Example 2:

Note that


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1 -0.8 -0.6 -0.4 -0.2
0.2 0.4 0.6 0.8 1
-1
-0.5
0.5
1
y
t



$i \times j = k \quad j \times k = i \quad k \times i = j \quad j \times i = -k \quad k \times j = -i \quad i \times k = -j$

Important Note:

$\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$

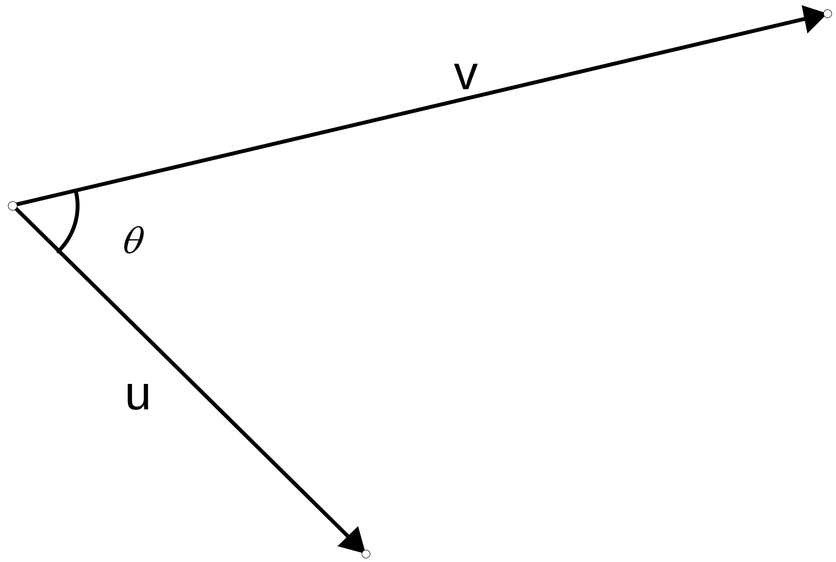
If we take

$\mathbf{u} \cdot (u \times v)$
$= \mathbf{u}_1(u_2v_3 - u_3v_2) + \mathbf{u}_2(u_3v_1 - u_1v_3) + \mathbf{u}_3(u_1v_2 - u_2v_1)$
$= 0$ **(check by distributing and cancelling the like terms)**

This means that u **is orthogonal to** $u \times v$

similarly, we can check that v **is orthogonal to** $u \times v$

Therefore $u \times v$ **is perpendicular to both** u **and** v



If the angle between u **and** v **is** θ**, where** $0 \le \theta \le \pi$

then we can relate θ **to** $u \times v$ **by using a few steps of algebra**

$\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$

$$\|u \times v\|^2$$
$$= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2$$
$$= \mathbf{u}_2^2\mathbf{v}_3^2 - 2\mathbf{u}_2\mathbf{v}_3\mathbf{u}_3\mathbf{v}_2 + \mathbf{u}_3^2\mathbf{v}_2^2 + \mathbf{u}_3^2\mathbf{v}_1^2 - 2\mathbf{u}_3\mathbf{v}_1\mathbf{u}_1\mathbf{v}_3 + \mathbf{u}_1^2\mathbf{v}_3^2 + \mathbf{u}_1^2\mathbf{v}_2^2 - 2\mathbf{u}_1\mathbf{v}_2\mathbf{u}_2\mathbf{v}_1 + \mathbf{u}_2^2\mathbf{v}_1^2$$
$$= \mathbf{u}_2^2\mathbf{v}_3^2 + \mathbf{u}_3^2\mathbf{v}_2^2 + \mathbf{u}_3^2\mathbf{v}_1^2 + \mathbf{u}_1^2\mathbf{v}_3^2 + \mathbf{u}_1^2\mathbf{v}_2^2 + \mathbf{u}_2^2\mathbf{v}_1^2 - 2\mathbf{u}_2\mathbf{v}_3\mathbf{u}_3\mathbf{v}_2 - 2\mathbf{u}_3\mathbf{v}_1\mathbf{u}_1\mathbf{v}_3 - 2\mathbf{u}_1\mathbf{v}_2\mathbf{u}_2\mathbf{v}_1$$
$$= \mathbf{u}_2^2\mathbf{v}_3^2 + \mathbf{u}_3^2\mathbf{v}_2^2 + \mathbf{u}_3^2\mathbf{v}_1^2 + \mathbf{u}_1^2\mathbf{v}_3^2 + \mathbf{u}_1^2\mathbf{v}_2^2 + \mathbf{u}_2^2\mathbf{v}_1^2 + \mathbf{u}_1^2\mathbf{v}_1^2 + \mathbf{u}_2^2\mathbf{v}_2^2 + \mathbf{u}_3^2\mathbf{v}_3^2 - 2\mathbf{u}_2\mathbf{v}_3\mathbf{u}_3\mathbf{v}_2 - 2\mathbf{u}_3\mathbf{v}_1\mathbf{u}_1\mathbf{v}_3 - 2\mathbf{u}_1\mathbf{v}_2\mathbf{u}_2\mathbf{v}_1 - \mathbf{u}_1^2\mathbf{v}_1^2 - \mathbf{u}_2^2\mathbf{v}_2^2 - \mathbf{u}_3^2\mathbf{v}_3^2$$
$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2$ **(Are you all still here?)**
$$= \|u\|^2\|v\|^2 - (u \cdot v)^2$$
$$= \|u\|^2\|v\|^2 - (\|u\|\|v\|\cos\theta)^2$$
$$= \|u\|^2\|v\|^2 - \|u\|^2\|v\|^2\cos^2\boldsymbol{\theta}$$

$$= \|u\|^2 \|v\|^2 (1 - \cos^2\theta)$$
$$= \|u\|^2 \|v\|^2 \sin^2\boldsymbol{\theta}$$

Since, $0 \le \theta \le \pi$**,** $\sin\theta \ge 0$

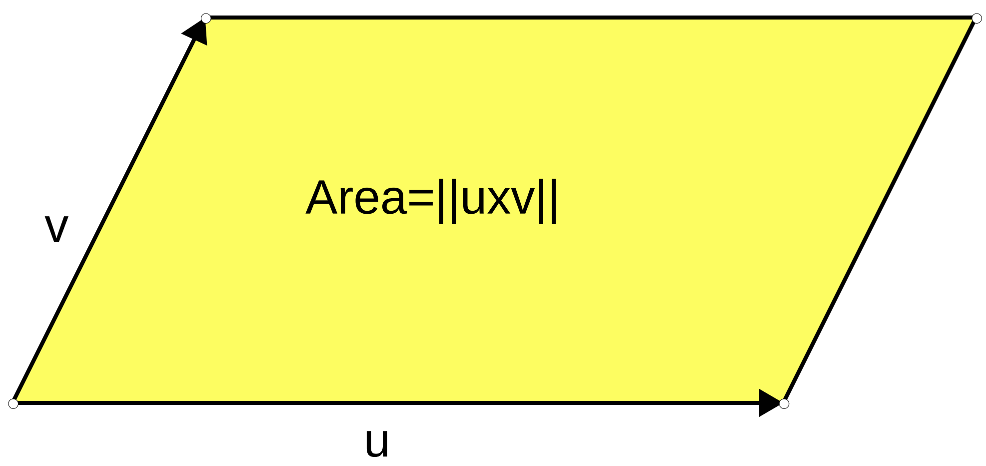
$$\boxed{\|u \times v\| = \|u\| \|v\| \sin\theta}$$

Also

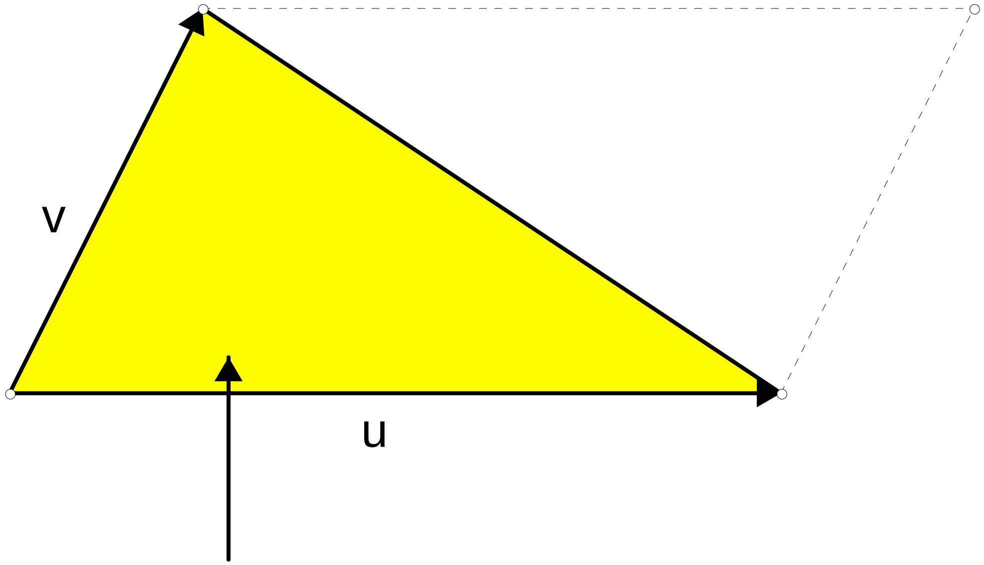
$\boxed{\text{If } u \text{ and } v \text{ are parallel, } u \times v = 0}$

We can apply this result to obtain the following

The area of a parallelogram



The area of a triangle



Area=(1/2)||uxv||

Note that a vector product is

NOT COMMUTATIVE, actually $\boxed{u \times v = -v \times u}$

NOT ASSOCIATIVE:

Consider:

u $=\langle 1,-1,2\rangle$ **v** $=\langle 2,1,1\rangle$ **w** $=\langle 1,-1,3\rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = (-1-2)\mathbf{i} + (4-1)\mathbf{j} + (1+2)\mathbf{k} = \mathbf{-3i} + \mathbf{3j} + \mathbf{3k}$$

$$(u\times v)\times\mathbf{w} = \begin{vmatrix} i & j & k \\ -3 & 3 & 3 \\ 1 & -1 & 3 \end{vmatrix} = (9+3)\mathbf{i} + (3+9)\mathbf{j} + (3-3)\mathbf{k} = \mathbf{12i} + \mathbf{12j}$$

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} = (1+3)\mathbf{i} + (1-6)\mathbf{j} + (-2-1)\mathbf{k} = \mathbf{4i} - \mathbf{5j} - \mathbf{3k}$$

$$\mathbf{u}\times(v\times w) = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 4 & -5 & -3 \end{vmatrix} = (3+10)\mathbf{i} + (8+3)\mathbf{j} + (-5+4)\mathbf{k} = \mathbf{13i} + \mathbf{11j} - \mathbf{k}$$

with these values

$(u\times v)\times\mathbf{w} \neq \mathbf{u}\times(v\times w)$

Scalar Triple Product:

If

$u =< u_1, u_2, u_3 >$, $v =< v_1, v_2, v_3 >$, $w =< w_1, w_2, w_3 >$

$\mathbf{v} \times \mathbf{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - w_1 v_3, v_1 w_2 - v_2 w_1 \rangle$

$\mathbf{u} \cdot (v \times w) = \mathbf{u}_1 (v_2 w_3 - v_3 w_2) + \mathbf{u}_2 (v_3 w_1 - w_1 v_3) + \mathbf{u}_3 (v_1 w_2 - v_2 w_1)$

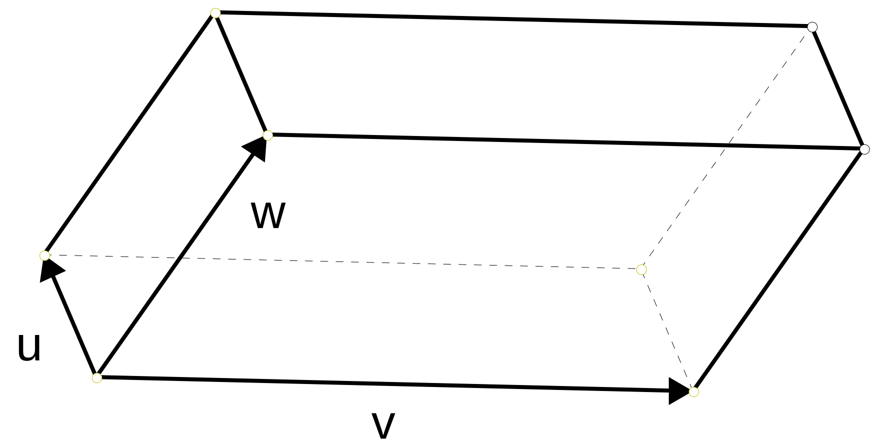
That is

$$\mathbf{u} \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

is called the scalar triple product of the vectors u, v, w

Application:

The volume of the parallelopiped shown below is $|u \cdot (v \times w)|$



Worked out Examples from the Text:

Section 11.4

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#10.

To find $u \times v$**,** $v \times u$**, and** $v \times v$

If

$u = \langle 3, -2, -2 \rangle$ **and** $v = \langle 1, 5, 1 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 3 & -2 & -2 \\ 1 & 5 & 1 \end{vmatrix} = (-2+10)\mathbf{i} + (-2-3)\mathbf{j} + (15+2)\mathbf{k} = \mathbf{8i} - \mathbf{5j} + \mathbf{17k}$$

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} i & j & k \\ 1 & 5 & 1 \\ 3 & -2 & -2 \end{vmatrix} = (-10+2)\mathbf{i} + (3+2)\mathbf{j} + (-2-15)\mathbf{k} = -\mathbf{8i} + \mathbf{5j} - \mathbf{17k}$$

Note that $u \times v = -v \times u$

For $v \times v$ **, note that the angle** θ **is** 0°

therefore

$\|v \times v\| = \|v\|^2 \sin \mathbf{0}^\circ = \mathbf{0}$

or you may verify this by the routine method shown above.

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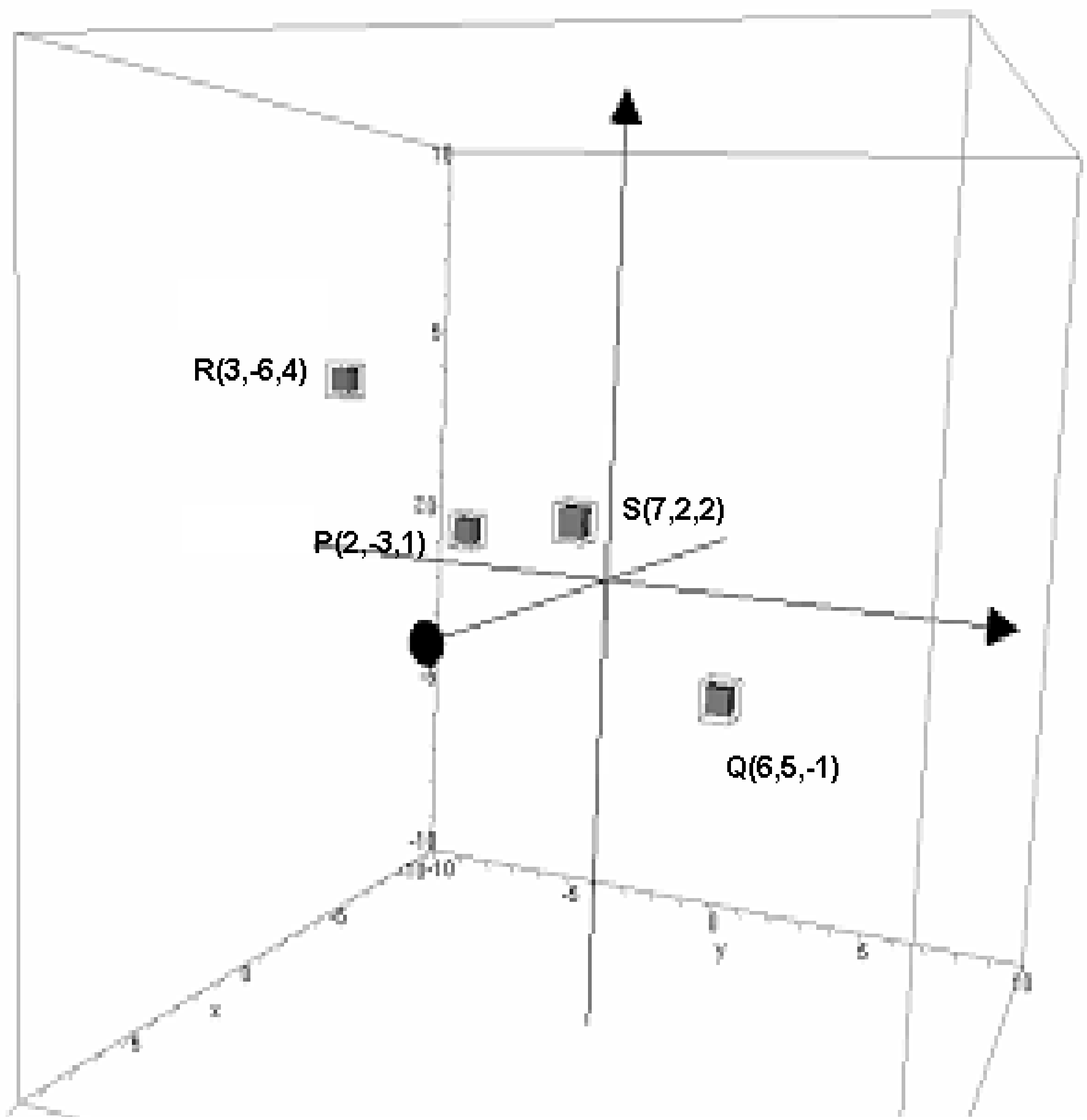
R(3,-6,4)
P(2,-3,1)
S(7,2,2)
Q(6,5,-1)

$\overrightarrow{PQ} = \langle 6-2, 5-(-3), -1-1\rangle = \langle 4, 8, -2\rangle$
$\overrightarrow{RS} = \langle 7-3, 2-(-6), 2-4\rangle = \langle 4, 8, -2\rangle$

The two vectors are quivalent, which means that these are parallel lines

$\overrightarrow{PR} = \langle 3-2, -6-(-3), 4-1\rangle = \langle 1, -3, 3\rangle$
$\overrightarrow{QS} = \langle 7-6, 2-5, 2-(-1)\rangle = \langle 1, -3, 3\rangle$

The two vectors are quivalent, which means that these are parallel lines

Therefore the figure is a parallelogram.

The area

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \left\|\begin{vmatrix} i & j & k \\ 4 & 8 & -2 \\ 1 & -3 & 3 \end{vmatrix}\right\|$$
$$= \|(24-6)i + (-2-12)j + (-12-8)k\|$$
$$= \|18i - 14j - 20k\|$$
$$= \sqrt{18^2 + (-14)^2 + (-20)^2}$$
$$= 2\sqrt{230}$$

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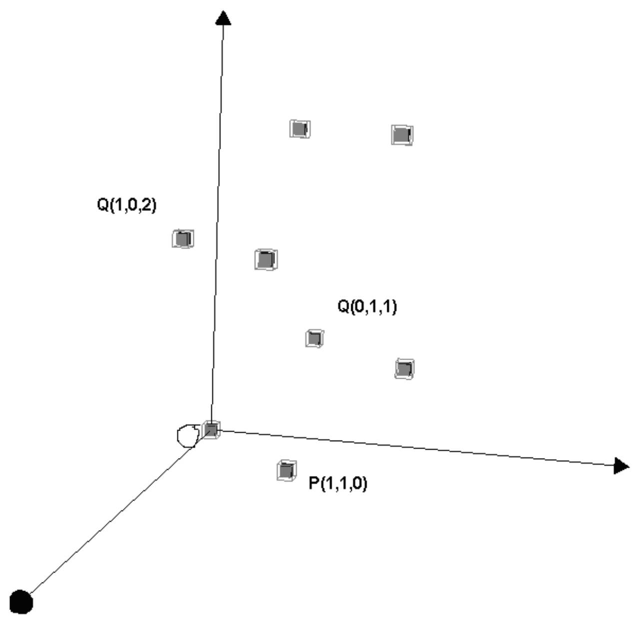
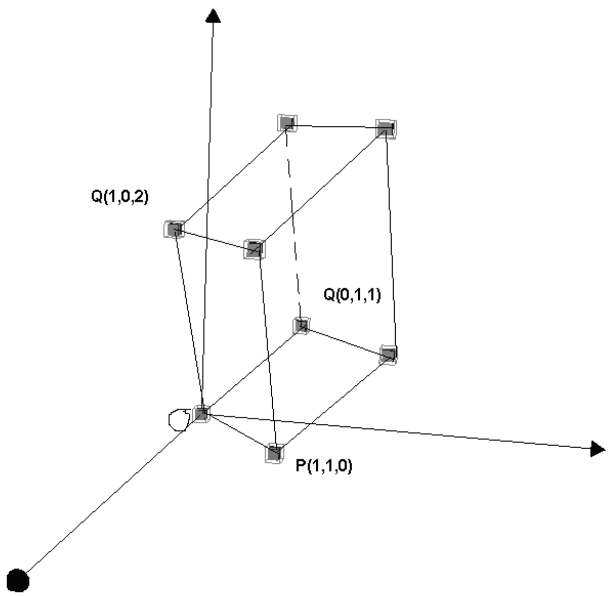
To find the volume,

we have to find vectors along three coterminous edges (the edges that terminate at one point)

and find the absolute value of the triple scalar product of these three vectors.





The volume is $\left|\overrightarrow{OP}\cdot\left(\overrightarrow{OQ}\times\overrightarrow{OR}\right)\right|$

$\overrightarrow{OP}=\langle 1,1,0\rangle$
$\overrightarrow{OQ}=\langle 0,1,1\rangle$
$\overrightarrow{OR}=\langle 1,0,2\rangle$

$$\left|\overrightarrow{OP}\cdot\left(\overrightarrow{OQ}\times\overrightarrow{OR}\right)\right|$$

=

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix}$$

$= \mathbf{3}$

Please work on (for practice)

#9,#15,#19,#29,#31,#35,#37,#43,#45,#47, #53 thru #64

If you have any difficulties, please post them in the discussion area.