Lesson 9:

This lesson will addess the section 1.8 in the third edition and section 1.7 in the second updated edition.

A function T: $R^n \to R^m$, assigns to each element of R^n a unique element of R^m. We shall also address the functions as mappings or transformations.

Example 1. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_2 + x_3 \end{bmatrix}$ is a function from $R^3 \to R^2$.

Example 2. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 + x_3 + x_4 \\ x_3 + 2x_4 \end{bmatrix}$ is a function from $R^4 \to R^3$.

Example 3. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \ln(x_1) \\ \ln(x_2) \end{bmatrix}$ can NOT be a function from $R^2 \to R^2$, because the negative coordinates will not have any real value.

Example 4. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \ln|x_1| \\ \ln|x_2| \end{bmatrix}$ is a function from $R^2 \to R^2$.

Example 5. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$ then $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a function $R^2 \to R^3$.

Make sure that you read the projection and shear transformations in the examples 2 and 3.

We shall be interested in special kinds of function called linear transformations.

Recall the definition of a Linear Transformation,
A function T: $R^n \to R^m$ is linear if:
(i) $T(u+v) = T(u) + T(v)$ for all u, v in R^n.
(ii) $T(\alpha u) = \alpha T(u)$ for all u in R^n and all real numbers α.

Note that particular, a linear transformation T: $R^n \to R^m$ must have the following propoerties.

T(0)=0

$T(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_p u_p) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \alpha_3 T(u_3) + \ldots + \alpha_p T(u_p)$

We shall with the techniques coming up soon that it is easy to verify that the examples 1,2 and 5 shown above are linear transformations but example 4 is not a linear transformation.

Let us take up some exercises from the text book.

I am taking the exercises from the third edition in a way that you should be able to follow even if you have the second updated edition.

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2 on page 79 (Third Edition)

Given the transformation T(x)=Ax with A= $\begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}$. To find T(u) and T(v) where

u= $\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ and v= $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

T(u)= $\begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \\ -2.0 \end{bmatrix}$

T(v)= $\begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.5a \\ 0.5b \\ 0.5c \end{bmatrix}$

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6 on page 80 (Third Edition)

Given that A= $\begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$, b= $\begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$ and T defined by T(x)=Ax.

To find x such that T(x)=b

Note that in this case x has the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

again, we want x = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that T$\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$

or a solution of the matrix equation

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$$

Look at the augmented matrix

$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 3 & -4 & 5 & 9 \\ 0 & 1 & 1 & 3 \\ -3 & 5 & -4 & -6 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since we have a free variable, there will be infinite number of solutions.

Therefore

$x_1 + 3x_3 = 7 \rightarrow x_1 = 7 - 3x_3$

$x_2 + x_3 = 3 \rightarrow x_2 = 3 - x_3$

Take $x_3 = 0$ or any other real number you would like,

$x_1 = 7$

$x_2 = 3$

Therefore one answer is x= $\begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$ note that $\begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$

Take $x_3 = 1$, $x_1 = 4$, $x_2 = 2$

another answer is x= $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ note that $\begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$

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10 on page 80 (Third Edition)

Given that A= $\begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -3 \\ 0 & 1 & 2 & 4 \\ -2 & 3 & 0 & 5 \end{bmatrix}$ and linear transformation T(x)=Ax, to find all x in R^4 such that T(x)=0.

Let x= $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in R^4$, Ax=0→ $\begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -3 \\ 0 & 1 & 2 & 4 \\ -2 & 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -3 & 0 \\ 0 & 1 & 2 & 4 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

x_3 is free variable and

$x_1 + 3x_3 = 0$
$x_2 + 2x_3 = 0$
$x_4 = 0$

$x_1 = -3x_3$
$x_2 = -2x_3$
$x_3 = x_3$
$x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$ where s can be any real number.

Note that

$$\begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -3 \\ 0 & 1 & 2 & 4 \\ -2 & 3 & 0 & 5 \end{bmatrix} \left(s \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right) = s \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -3 \\ 0 & 1 & 2 & 4 \\ -2 & 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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12 on page 80 (Third Edition)

Given that A= $\begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -3 \\ 0 & 1 & 2 & 4 \\ -2 & 3 & 0 & 5 \end{bmatrix}$ and linear transformation T(x)=Ax, b= $\begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$

To see if b is in the range of T that is if we can find $x \in R^4$ such that Ax=b

If x= $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Ax=b

$$\rightarrow \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -3 \\ 0 & 1 & 2 & 4 \\ -2 & 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$$

Look at the augmented matrix

$$\begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -3 & 3 \\ 0 & 1 & 2 & 4 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the last row has non zero entry only in the augmented column, the system has no solution, therefore no such x is available.

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20 on page 80 (Third Edition)

So far we were given a matrix defining the map, in this problem we are given a linear transformation, and we have to obtain A.

Given that x= $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $v_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, $v_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

T: $R^2 \rightarrow R^2$ is defined by T(x)= $x_1 v_1 + x_2 v_2$. To find A such that T(x)=Ax

T(x)= $x_1 v_1 + x_2 v_2$

$$\rightarrow x_1 \begin{bmatrix} -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}.$$

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32 on page 81 (Third Edition)

32.
To show that $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

Note that

$T((4,1) + (2,-2)) = T(6,-1) = (4 \times 6 - 2 \times (-1), 3|-1|) = (26,3)$
$T(4,1) + T(2,-2) = (4 \times 4 - 2 \times 1, 3|1|) + (4 \times 2 - 2 \times (-2), 3|-2|) = (26,9)$

Therefore $T((4,1) + (2,-2)) \neq T(4,1) + T(2,-2)$

and the map is NOT linear.

Please work on the exercises in the section 1.8 (Third Edition) and section 1.7 in the second updated edition.