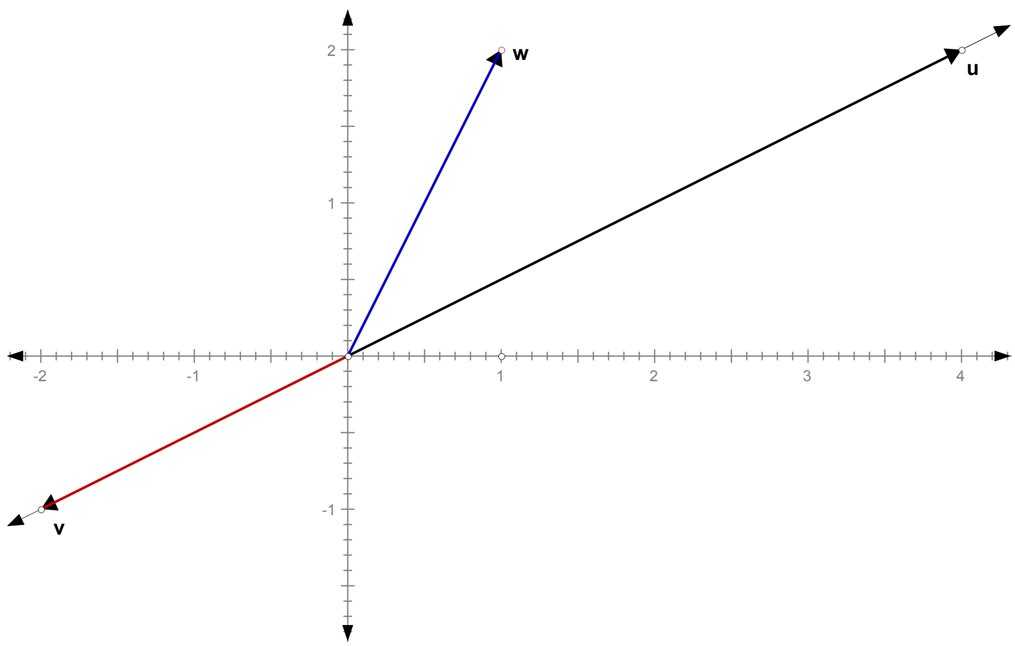
Lesson 8:

Before starting to read the following text ,
first read the Definition on page 65 in the Third Edition **or**
the Definition on page 59 in the second updated edition

Example 1.



Look at the vectors u= $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$**, v=** $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$**, w=** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Note that the vectors u and v are along the same straight line or are collinear, while w is not collinear with u or w.

Let us understand the algebraic aspect of such relationships.
u and v are collinear,
and note that

u=2v
or
u−2v=0

That is we have a linear relation $\alpha_1 u + \alpha_2 v = 0$ with not all of α_1 and α_2 zeros
because $\alpha_1 = 1$, and $\alpha_2 = -2$.
When such a relationship is satisfied, we call the set {u,v} of vectors as linearly dependent.

On the other hand, if we look at the set {u,w}
then a relationship like $\alpha_1 u + \alpha_2 w = 0$ with not all of α_1 and α_2 zeros **is not possible,**
because

$$\alpha_1 u + \alpha_2 w = 0 \rightarrow \alpha_1 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Look at the augmented matrix $\begin{bmatrix} 4 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow$

$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **is the only possibility**
for a relationship like $\alpha_1 u + \alpha_2 w = 0$.

Therefore we say that {u,w} are linearly independent.

However

If we took {u,v,w} as a set of vectors, then because of u−2v=0, we can write that u−2v+0w=0, that is have a relationship
α_1u+α_2v+α_3w=0 with not all the α_1,α_2,α_3 zeros, sometime we shall refer to this as not all αs are zeros.

In view of this we see that {u,v,w} is a linearly dependent set of vectors.

You can prove that any set that has a linearly dependent subset is linearly dependent.

Example 2.

In R^3

consider the subset $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ **.**

To check if these three vectors are linearly dependent.

To see if it is possible to get $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **such that**

$$\mathbf{x}_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mathbf{x}_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \mathbf{x}_3 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This shows that x_3 **is a free variable, and**

$\mathbf{x}_1 - \mathbf{x}_3 = \mathbf{0} \rightarrow \mathbf{x}_1 = \mathbf{x}_3$

$\mathbf{x}_2 + \mathbf{2x}_3 = \mathbf{0} \rightarrow \mathbf{x}_2 = -\mathbf{2x}_3$

choose $x_3 = 1$ **or any other real number different from 0, and see**

$\mathbf{x}_1 = \mathbf{1}$ $\quad \mathbf{x}_2 = -\mathbf{2}$ $\quad \mathbf{x}_3 = \mathbf{1}$

Note that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \mathbf{2} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

therefore $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ **is a linearly depenent subset of** R^3.

Example 3:

Consider the subset S= $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ **of** R^3**.**

For the equation

$$\mathbf{x}_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mathbf{x}_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \mathbf{x}_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ldots\ldots(\mathbf{1})$$

note $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

which implies that the only way the equation (1) will hold is $x_1 = x_2 = x_3 = 0$ **therefore S is a linearly independent subset of** R^3**.**

Look at the following examples (4 and 5) in the light of the fact (3) cited on the page 60 in the second updated edition or page 66 in the third edition.

Example 4:

To see if the columns of A= $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 3 & 1 & 2 & 1 \end{bmatrix}$ **are linearly independent, all we need to do is to see whether all the columns have pivot positions.**

Since $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 3 & 1 & 2 & 1 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$**, all the colmns have pivot positions, and the columns are linearly independent.**

Example 5:

To see if the columns of A= $\begin{bmatrix} 1 & 2 & 1 & 5 \\ 2 & 1 & 1 & 6 \\ 1 & 1 & 1 & 4 \\ 3 & 1 & 2 & 9 \end{bmatrix}$ **are linearly independent, all we need to do is to see whether all the columns have pivot positions.**

Since, $\begin{bmatrix} 1 & 2 & 1 & 5 \\ 2 & 1 & 1 & 6 \\ 1 & 1 & 1 & 4 \\ 3 & 1 & 2 & 9 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$**, the fourth column does not have any pivot position, and the columns are not linearly dependent that is they are linearly independent.**

Example 6:

To check if $\left\{ \begin{bmatrix} -4 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 10 \\ -6 \\ -8 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 5 \end{bmatrix} \right\}$ **is linearly independent.**

Note that the vectors are the columns of the matrix, A= $\begin{bmatrix} -4 & 10 & -5 & -6 \\ 7 & -6 & 7 & 0 \\ 8 & -8 & 6 & 5 \end{bmatrix}$ **and the row reduced echelon form of this matrix can NOT have pivot position in each column as there are only three rows, therefore the given subset will have to be linearly depenedent.**

This technique is mentioned in the Theorem 8 on the page 69 of the third edition or the Theorem 8 on the page 62 of the second updated edition.

Example 7:

Consider $v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 2 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$

We can note that $v_1 + v_2 = v_3$ **that is** v_3 **is a linear combination of** $v_1, v_2.$ **therefore the subset** $\{v_1, v_2, v_3, v_4\}$ **is linearly dependent in the light of earlier discussions or more specifically, Theorem 7 on page 68 (third edition) or Theorem 7 on the page 61 of the second edition.**

NOTE that even though v_4 **can not be written as a linear combination of the other vectors still the subset is linearly dependent.**

Since the zero vector in R^n, **can be written as a linear combiantion of any set of vectors in** R^n **, any subset containg the zero vector is linearly dependent.**

This is the result stated in the Theorem 9 (section 1.7 third edition and section 1.6 second updated edition.)

Please work on the exercises in section 1.7 in the third edition or 1.6 in the second updated edition.