Lesson 7:

In this lesson, we shall discuss how to describe the solutiuon set of a system of linear equations.

First some terminologies:

If b=0, we say that the system of linear equations Ax=b is homogeneous**.**

Example 1:

$$x_1 - 2x_2 + x_3 - x_4 = 0$$
$$x_1 - x_2 + x_3 - 4x_4 = 0$$
$$2x_1 + 2x_2 - x_3 - 3x_4 = 0$$
$$x_1 - 2x_2 + 2x_3 - x_4 = 0$$

or

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & -1 & 1 & -4 \\ 2 & 1 & -1 & -3 \\ 1 & -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that this system is always consistent because $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ **is always a solution, called** TRIVIAL SOLUTION**, a solution that is different from this is a** NON TRIVIAL SOLUTION**.**

Remember that a system may have more than one solution, depending on the number of free variables,
let us look at the augmented matrix of the above system

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 0 \\ 1 & -1 & 1 & -4 & 0 \\ 2 & 1 & -1 & -3 & 0 \\ 1 & -2 & 2 & -1 & 0 \end{bmatrix}$$ **, row echelon form:** $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The first four columns, that correspond to the variables in the system, all have pivot positions**, therefore there are no free variables,**

and note that

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

shows that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ **is the only solution.**

Example 2:

Consider the system

$x_1 + x_2 + 2x_3 - 3x_4 = 0$

$2x_1 - x_2 + x_3 + 9x_4 = 0$

$-x_1 + 2x_2 + x_3 - 12x_4 = 0$

$x_1 + x_2 + 2x_3 - 3x_4 = 0$

or

$$\begin{bmatrix} 1 & 1 & 2 & -3 \\ 2 & -1 & 1 & 9 \\ -1 & 2 & 1 & -12 \\ 1 & 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Look at

$$\begin{bmatrix} 1 & 1 & 2 & -3 & 0 \\ 2 & -1 & 1 & 9 & 0 \\ -1 & 2 & 1 & -12 & 0 \\ 1 & 1 & 2 & -3 & 0 \end{bmatrix}$$ **, row echelon form:** $$\begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the variables x_3, and x_4 are free therefore there are infinite number of solutions.

The above means that
$\mathbf{x}_1+\mathbf{x}_3+2\mathbf{x}_4= 0$
$\mathbf{x}_2+\mathbf{x}_3-5\mathbf{x}_4= 0$

First write the basic variables in terms of the free variables, i.e. move the free variables to the right hand side
$\mathbf{x}_1= -\mathbf{x}_3-2\mathbf{x}_4$
$\mathbf{x}_2= -\mathbf{x}_3+5\mathbf{x}_4$

Extend the above with equations describing all the variables (basic as well as free) in terms of the free variables only.

That is

$\mathbf{x}_1= -\mathbf{x}_3-2\mathbf{x}_4$
$\mathbf{x}_2= -\mathbf{x}_3+5\mathbf{x}_4$
$\mathbf{x}_3= \mathbf{x}_3+0\mathbf{x}_4$
$\mathbf{x}_4= 0\mathbf{x}_3+\mathbf{x}_4$

Which may be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{x}_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \mathbf{x}_4 \begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

where x_3, x_4 can be given any real values.

or rewrite it as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

where s and t can be given any real values.

An illustration is

take $s = 4$ **and** $t = -3$ **(a set of arbitrtay values)**

$$4\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} -3\begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -19 \\ 4 \\ -3 \end{bmatrix}$$

therefore $\begin{bmatrix} 2 \\ -19 \\ 4 \\ -3 \end{bmatrix}$ **is a non trivial solution of** $\begin{bmatrix} 1 & 1 & 2 & -3 \\ 2 & -1 & 1 & 9 \\ -1 & 2 & 1 & -12 \\ 1 & 1 & 2 & -3 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

because $\begin{bmatrix} 1 & 1 & 2 & -3 \\ 2 & -1 & 1 & 9 \\ -1 & 2 & 1 & -12 \\ 1 & 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -19 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Example 3:

$x_1 + x_2 + 2x_3 - 3x_4 = 5$

$2x_1 - x_2 + x_3 + 9x_4 = -11$
$-x_1 + 2x_2 + x_3 - 12x_4 = 16$
$x_1 + x_2 + 2x_3 - 3x_4 = 5$
Consider

$$\begin{bmatrix} 1 & 1 & 2 & -3 \\ 2 & -1 & 1 & 9 \\ -1 & 2 & 1 & -12 \\ 1 & 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{matrix} 5 \\ -11 \\ 16 \\ 5 \end{matrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & -3 & 5 \\ 2 & -1 & 1 & 9 & -11 \\ -1 & 2 & 1 & -12 & 16 \\ 1 & 1 & 2 & -3 & 5 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & 1 & 2 & -2 \\ 0 & 1 & 1 & -5 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which means that

$x_1 + x_3 + 2x_4 = -2$
$x_2 + x_3 - 5x_4 = 7$

Write the all the variables, in terms of the free variables and the values in the constant columns.

$x_1 = -2 - x_3 - 2x_4$
$x_2 = 7 - x_3 + 5x_4$
$x_3 = 0 + x_3 + 0x_4$
$x_4 = 0 + 0x_3 + x_4$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 0 \\ 0 \end{bmatrix} + \mathbf{x}_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \mathbf{x}_4 \begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

or you may rephrase the above as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 0 \\ 0 \end{bmatrix} + \mathbf{s} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \mathbf{t} \begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

where s and t can be given any values.

For example:

Take s=**1, t**=-3

$$\begin{bmatrix} -2 \\ 7 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 1 \\ -3 \end{bmatrix}$$

check:

$$\begin{bmatrix} 1 & 1 & 2 & -3 \\ 2 & -1 & 1 & 9 \\ -1 & 2 & 1 & -12 \\ 1 & 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -11 \\ 16 \\ 5 \end{bmatrix}$$ **the right hand side of the given system.**

Moreover, note that

the system in the example 2 is the same system that you have in the example 3 with the right hand side replaced by 0.

If we took a solution from example 2 of Ax=0, say $\begin{bmatrix} 2 \\ -19 \\ 4 \\ -3 \end{bmatrix} = v_h$ **(notation, h for homogeneous) and a particular solution from example 3, say** $\begin{bmatrix} 3 \\ -9 \\ 1 \\ -3 \end{bmatrix} = p$ **(notation for a particular solution), then** $v_h + p$ **is also a solution of the system in the example 3.**

Note that

$$\begin{bmatrix} 1 & 1 & 2 & -3 \\ 2 & -1 & 1 & 9 \\ -1 & 2 & 1 & -12 \\ 1 & 1 & 2 & -3 \end{bmatrix} \left(\begin{bmatrix} 2 \\ -19 \\ 4 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ -9 \\ 1 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -11 \\ 16 \\ 5 \end{bmatrix}$$

the right hand side of the given system in the example 3.

Read Theorem 6 in section 1.5 oif both the editions in this light.

Also read

The instructions for writing the solutions on

Page 53 of the Second Updated Edition

or

Page 54 of the Third Edition

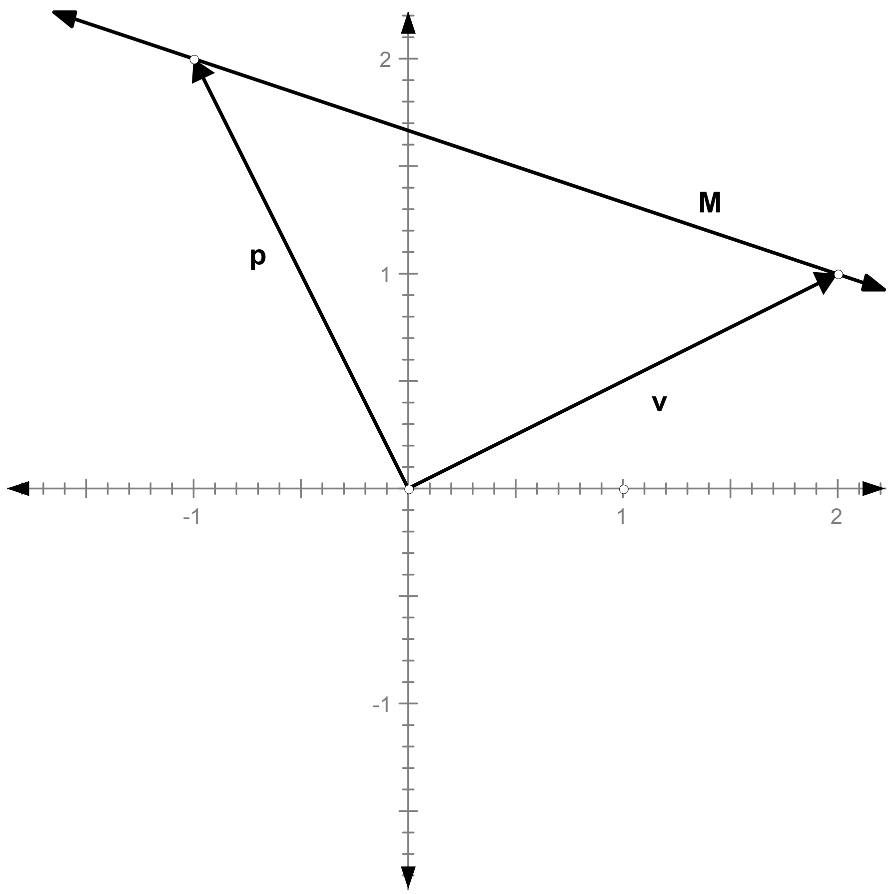
Read the introduction of equation of a line on page 51 in the second edition or page 53 in the third edition.

Writing Parametric Equation of a Line passing through two vectors.

Example 4:

Write parametric equation of the line (L) passing through $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **and**

$p = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$



Note that M is parallel to $v - p = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Therefore a parametric equation is

x=v+t$(v - p)$

or

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mathbf{t} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

You are ready to try exercises in the sections 1.4 and 1.5.