Please recall the Theorem 4 from the section 1.4:

It reads

Let A be an mxn matrix. Then the following are logically equivalent. That is, for a particular A, either all the following statements are true or they are all false.

a. For each b in R^m, the equation Ax=b has a solution.
b. Each b in R^m is a linear combination of the columns of A.
c. The columns of A span R^m.
d. A has a pivot position in each row.

Most of the solutions in the following exercises from the text involve the use of this theorem. Note that even if you have the second edition, you should still be able to follow/

Exercise 16 on page 48 (Third Edition)

Given that $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ **and** $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$**.**

a) To show that Ax=b does not have a solution for all possible b that is part a) of the theorem 4 above is not true in this example.
b) To describe the set of all b for which Ax=b has a solution.

a) If we have to answer only the part a), we may look at
$\begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & -\frac{10}{7} \\ 0 & 1 & \frac{6}{7} \\ 0 & 0 & 0 \end{bmatrix}$ **and note that the third row does not have any pivot position,**
therefore part d) of the theorem is not true, which shows that part a) will not be true either.

b) Note that we could have combined a) and b)

Look at

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix} \xrightarrow[r_3 - 5r_1]{r_2 + 3r_1} \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 14 & 12 & b_3 - 5b_1 \end{bmatrix}$$

at this stage note the

relationship between the second and the third row,

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2+3b_1 \\ 0 & 14 & 12 & b_3-5b_1 \end{bmatrix} \xrightarrow{r_3+2r_2} \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2+3b_1 \\ 0 & 0 & 0 & b_1+2b_2+b_3 \end{bmatrix}$$

for a solution, we must have
$b_1 + 2b_2 + b_3 = 0$ **we are done with the solution of the problem.**

just to ADD to the solution

to verify this note that
$b_1 = 1, b_2 = 1, b_3 = -3$ **satisfies the above equation. These are just one set of values that you can take.**

With these values
$$\begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 2 & 6 & 1 \\ 5 & -1 & -8 & -3 \end{bmatrix}, \textbf{ row echelon form: } \begin{bmatrix} 1 & 0 & -\frac{10}{7} & -\frac{5}{7} \\ 0 & 1 & \frac{6}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
Therefore a solution is available.

Exercise 18 on page 48 (Third Edition)

i) To verify if the columns of B=$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$ **span** R^4 **.**

ii) To verify whether Bx=y have a solution for each y in R^4 **.**

part i)

Note that part i) relates to part c) of the theorem cited above. since part c) is related to part d) all we have to do is to row reduce B.

B= $\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$ **, row echelon form:** $\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Note that the last row does not have any pivot position. Therefore d) is not true which means
c) can not be true and the columns of A do not span R^4**.**

part ii) Note that this statement is related to part a) of the Theorwm 4 above. Since we just noted both parts c) and d) are note true,
therefore a) can not be true either and Bx=y DOES NOT have a solution for each y in R^4 **with this value of B.**

We are done with the answer.

An extended note:

Part ii) simply states that Bx=y DOES NOT have a solution for each **y in** R^4

which means that you may find some y= $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in R^4$ **for which Bx=y will have a solution.**

You may work on $\begin{bmatrix} 1 & 3 & -2 & 2 & a \\ 0 & 1 & 1 & -5 & b \\ 1 & 2 & -3 & 7 & c \\ -2 & -8 & 2 & -1 & d \end{bmatrix}$ **to find conditions on a,b,c,d so that Bx=y has a solution for such values.**

Exercise 20 on page 40 (Third Edition)

There are two parts to the question

i) Can every vector in R^4 **can be written as a linear combination of the columns of**

B= $\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$?

ii) Do the columns of B span R^3.

Part i) : the statement is part b) of the theorem 4 cited above. For B, we saw that neither of a,c or d is true therefore b) can not be true,

and the answer to the question is NO. Notice that even if we see one of the parts of the theorem 4 false, all the others will be false.

Part ii) The columns of B are NOT vectors in R^3, therefore they can NOT span R^3.

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True False question

c) The solution set of the linear system whose augmented matrix is

$\begin{bmatrix} a_1 & a_2 & a_3 & b \end{bmatrix}$

is the same as the soltuion set of Ax=b if A=$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$.

True : reason theorem 3 in section 1.4

d) If the equation Ax=b is inconsistent, then b is not in the set spanned by the columns of A.

True: because the first part (If part) says that part a) of the theorem 4 is False, therefore

part b) will be false as well.

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Let $u = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}, v = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, **and** $w = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$**. It can be shown that** $3u - 5v - w = 0$.

To use this fact , and NOT row operations to find $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **which satisfy the**

equation

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{3u - 5v - w = 0 \rightarrow 3u - 5v = w \rightarrow 3} \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} \mathbf{-5} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

showing that $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ **is a solution.**