Lesson 17:

Let us continue with our study of the section 4.2

Let V and W be vector spaces and T:V→W a linear transformation.
Recall that Kernel of T, denoted by KerT$=\{$v ∈ V: T(v)=0$\}$

Note that if T:$R^n \to R^m$ is a linear transformation and A is a standard matrix of this linear transformation,
then KerT=NulA

Facts about KerT.

1. Let V and W be vector spaces and T:V→W a linear transformation, then KerT is a subspace of V.

Proof:
i) 0∈KerT because T(0)=0 by the definition of KerT.
ii) u,v∈KerT ⇒ T(u)=T(v)=0 ⇒ T(u+v)=T(u)+T(v)=0+0=0 ⇒ u+v∈KerT
iii) For any real number α and any u∈KerT ⇒ T(cu)=cT(u)=c0=0 ⇒ cu∈KerT.
therefore all the requirements for being a subspace are satisfied and KerT is a subspace of V.

2. Let V and W be vector spaces and T:V→W a linear transformation. T is one-one ⇔ KerT={0}

Proof:

⇒

Given that T is one-one (that is T(u)=(v) → u=v)

$u \in KerT \rightarrow T(u)=0$
since $T(0)=0$

we have $T(u)=T(0)$. Since T is one-one, $u=0$.

therefore

$u \in KerT \rightarrow \quad u=0$

which means $KerT=\{0\}$

$\Leftarrow$

Now we are assuming that $KerT=\{0\}$ and have to show that
Given that T is one-one (that is $T(u)=(v) \quad \rightarrow \quad u=v$)

$T(u)=T(v) \rightarrow T(u)-T(v)=0 \rightarrow T(u-v)=0 \quad \rightarrow \quad u-v \in KerT$
($T(u)-T(v)=T(u-v)$ because T is a linear transformation.)
Since $KerT=\{0\}$ and $u-v \in KerT$, $u-v=0$ or $u=v$.

Therefore, we have shown that
$T(u)=T(v) \quad \rightarrow \quad u=v$.

Which means that T is one-one.

QED

Recall the definition of the range of a map on the page 232 in third edition or page 227 of the second updated edition.

Note the following property of the range as stated in the problem #30 on the page 235 of the third edition.

Let V and W be vector spaces and $T:V\rightarrow W$ a linear transformation, then Range of T is a subspace of W.

Proof:

a) $0\in$Range of T because $T(0)=0$ for 0 in V.
b) if p,q are in Range of T, we can find u and v in V such that $T(u)=p$ and $T(v)=q$.
$u+v \in V$, therefore $T(u+v)=T(u)+T(v)=p+q$ is in the Range of T.
c) Take a real number c and a vector w in the Range of T. We can find x in V such that $T(x)=w$.
$cx\in V$, therefore $T(cx)=cT(x)=cw$ is in the Range of T.

Therefore Range of T is a subspace of W.

Example 18.1

Let us take up #32 on page 235 in third edition.

$T:P_2 \rightarrow R^2$ is defined by $T(p)=\begin{bmatrix} p(0) \\ p(0) \end{bmatrix}$

We have to find
a) polynomials p_1 and p_2 in P_2 such that $KerT=Span\{p_1,p_2\}$
b) description of the Range of T.

a)
Consider $p=a+bx+cx^2 \in P_2$
$p=a+bx+cx^2 \in KerT \rightarrow p(0)=0 \rightarrow a=0$
therefore an element of KerT is of the form $bx+cx^2$ and

KerT=span$\{x, x^2\}$

b)

$p=a+bx+cx^2 \in P_2$

$\Rightarrow T(p)=\begin{bmatrix} a \\ a \end{bmatrix} \in$ range of T.

Note that for any real number a , $\begin{bmatrix} a \\ a \end{bmatrix} \in$ range of T.

Therefore Range of T=span$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

Example 18.2:

Look at #34 in the third edition.

We are given a map T: C[0,1] $\rightarrow$ C[0,1]
defined by T(f)= the antiderivative of F of f such that F(0)=0

a) we have to show that T is a linear transformation.
b) we have to descrobe KerT.

(Let us first see some examples of how does this map work.
Note that $f(x)=x^2+x \in C[0,1]$ and $F=\int(x^2+x)dx = \frac{x^3}{3}+\frac{x^2}{2}+C$ where C is an arbitrary constant.
according to the above definition $T(f)=\frac{x^3}{3}+\frac{x^2}{2}$

whereas for f(x)=sinx, note that $\int f(x)dx=-\cos(x)+C$, $F(x)=-\cos(x)+1$)

a) consider f,g in C[0,1]

T(f)=F

T(g)=G

F(0)=0

$G(0)=0$
therefore
$(F+G)(0)=F(0)+G(0)=0+0=0$

also for any real number α, $\alpha F(0)=0$

and

$T(f+g)=F+G=T(f)+T(g)$
$T(\alpha f)=\alpha F$

therefore T is a linear transformation.

b)

$f\in KerT \rightarrow$ the antiderivative F such that $F(0)=0$ will have to be the zero function.

Note that the zero function (a function with 0 as the only value) is a constant function.
Therefore the only possible definition of f is $f(x)=0$ for all x.

Or informally $f=0$

and $KerT=\{0\}$

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please finish all the practice problems in the section 4.2.