Lesson 13:

Before reading this lesson, please make sure that you have read, the definitions and the theorems in the section 2.9 of the revised third edition or the entire section 2.8 in the second updated edition.

Example 13.1:

Recall that S= $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of R^2 beacuse S spans R^2 and S is linearly independent.

Since, S is a basis, each element of R^2 can be uniquely written as a linear combination of the elements of S.

For example, if we took, $\begin{bmatrix} 6 \\ -2 \end{bmatrix} \in R^2$,

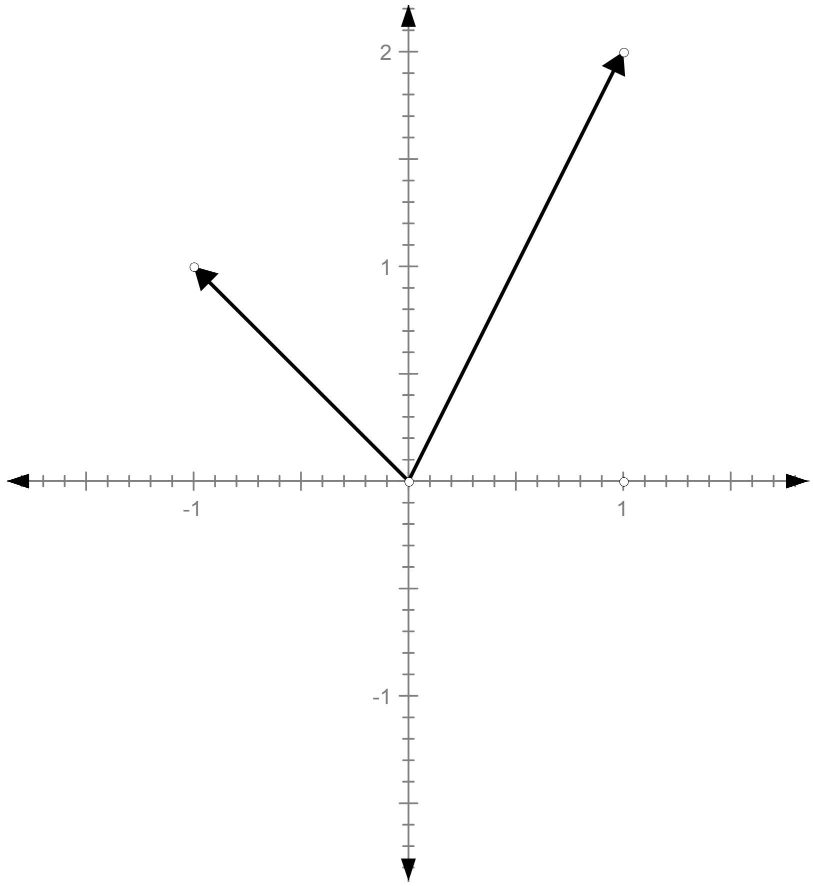
$\begin{bmatrix} 6 \\ -2 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

therefore 6 and -2 are the coordinates of $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$ with respect to the basis S.

This basis S is called the standard basis of R^2.

Example 13.2:

Look at a different subset B= $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ of R^2 as shown below.



First, note that

$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

therefore the vectors in B are linearly independent (both the columns are pivot columns) and span R^2 (all the rows have pivot positions.) Therefore each vector in R^2 can be uniquely written as a linear combination of the vectors in B.

For example take $\begin{bmatrix} -2 \\ 3 \end{bmatrix} \in R^2$

$\begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{7}{3} \end{bmatrix}$

shows that

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \tfrac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \tfrac{7}{3}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

therefore, $\frac{1}{3}$ and $\frac{7}{3}$ are the coordinates of $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ with respect to the basis B.

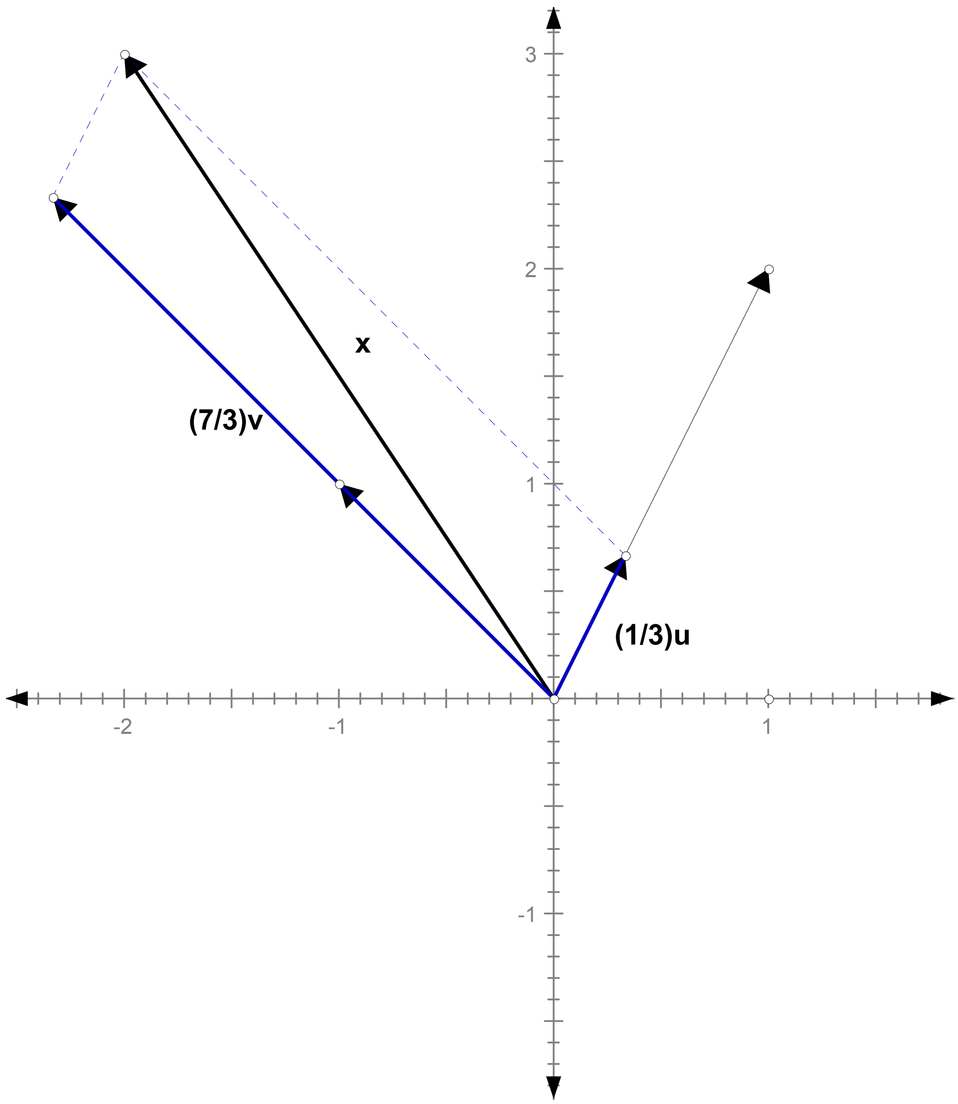
We write this information as

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix}_B = \begin{bmatrix} \frac{1}{3} \\ \frac{7}{3} \end{bmatrix}$$

using the subscript to identify the basis, unless the basis is the standard basis.

The following graph shows this with

$$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$



x=$\frac{1}{3}$u+$\frac{7}{3}$v

Example 13.3:

Let us work on the exercise 6 on the page 181 in the third edition, which gives us

x∈H, where B=$\{b_1, b_2\}$ is a basis of H with

x=$\begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix}, b_1 = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, b_2 = \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix}$,

we have to find $[x]_B$

All that we have to do is to see if the augmented matrix

$\begin{bmatrix} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{bmatrix}$ corresponds to a consistent system.

Since $\begin{bmatrix} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & -\frac{5}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

$[x]_B = \begin{bmatrix} -\frac{5}{2} \\ \frac{1}{2} \end{bmatrix}$

and note that

$$\begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix} = -\frac{5}{2}\begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix}$$

Example 13.4:

Exercise #10 on the page 181 in the third edition.
To find bases for NulA and ColA and state their dimensions, and we are given that

$\begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix}$, an echelon form is $\begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

It is a good idea to transform it to the row_reduced echelon form

$$\begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{r_1 + 2r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & 3 & 5 & -10 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{r_1 - 5r_3}{\rightarrow} \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Looking at the row reduced echelon form, note that if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \text{NulA}$$

then

$x_1 = -3x_3$

$x_2 = 3x_3 + 7x_5$

$x_4 = 2x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Therefore a basis of NulA is $\left\{ \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ and the

dimension of NulA is 2.

For a basis of the ColA, identify the pivot columns, which according to the row-reduced form are
the first, second, and fourth columns.Be careful to select these columns from the original matrix.

A basis for the colA is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix} \right\}$ and therefore

the dimension of the column space is 3.

Remember that the dimension of ColA is also called rank of A or rankA.

Example 13.5:

Let us take up #14 on the page 181 in the third edition, where have to find a basis for the subspace spanned by

$$\begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -7 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 9 \\ -5 \end{bmatrix}.$$

Note that the above subspace is the same as ColA with

$A=\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & 2 & 4 & -8 \\ -2 & -1 & -6 & -7 & 9 \\ 5 & 6 & 8 & 7 & -5 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 4 & 5 & -7 \\ 0 & 1 & -2 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Therefore a basis is $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix} \right\}$

Please make sure that you finish the chapter 2, and post the difficulties in the discussion area.