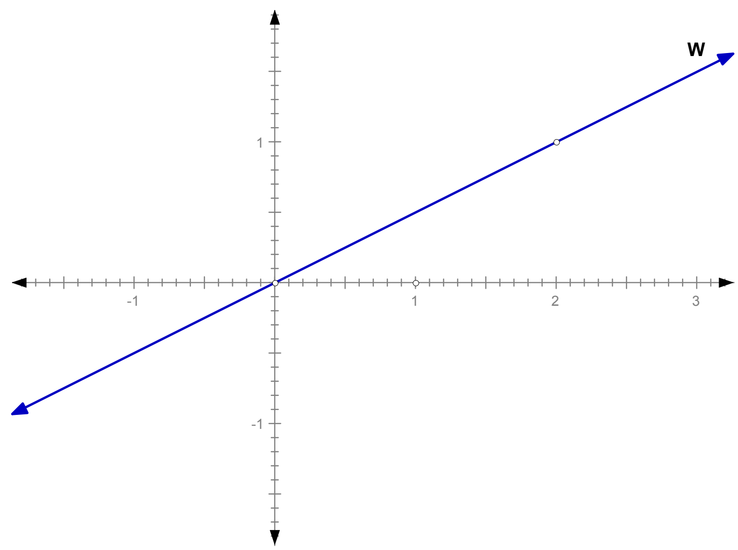
Lesson 12:

Before reading this lesson,
read the definitions of a subspace of R^n
the definition of the column space of a matrix
the definition of the null space of a matrix

We continue with some examples of the above
Example 12.1: Note that the subset W of R^2 shown in the picture 12.1, is a subspace of R^2 , because

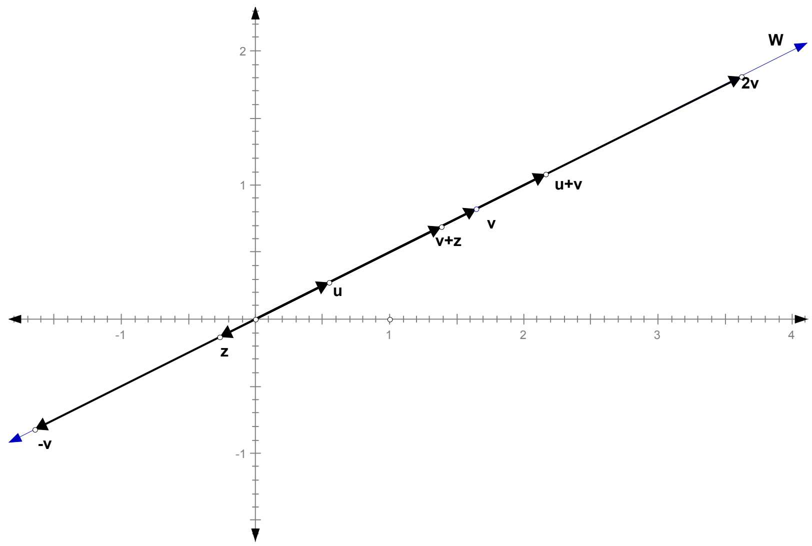


Picture 12.1

i) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in$W

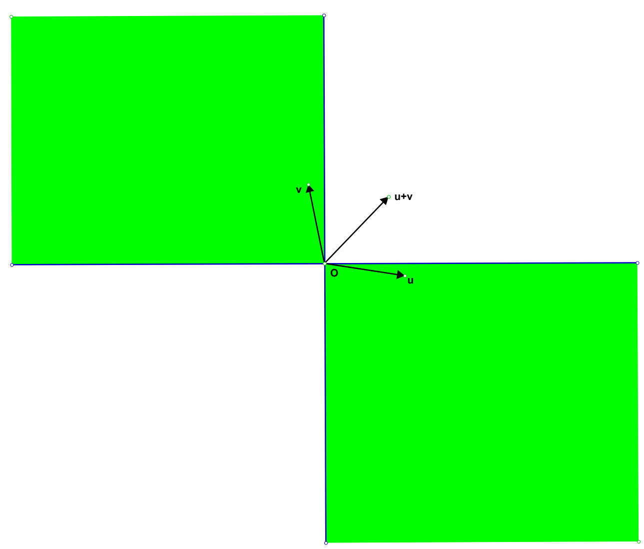
ii) sum of any two vectors in W, is in W. (picture 12.2)

iii) scalar multiple of a vector in W, is in W. (picture 12.2)



picture 12.2

Example 2:
If we take W to be the union of the two green regions, it is not a subspace of R^2 because it is not closed with respect to addition, that is u and v are vectors in W but u+v does not belong to W.



Example 12.3:. The plane W= $\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}\right\} \subset R^3$ can be easily seen to be a subspace of R^3. The plane is shown in the picture 12.3.

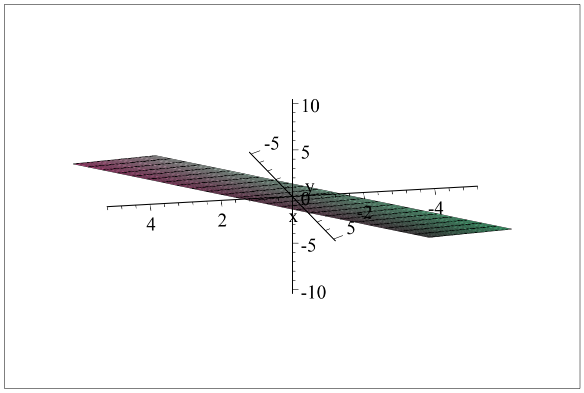


Figure 12.3

Example12.4:

The subset W= $\left\{ \begin{bmatrix} x \\ y \\ 2 \end{bmatrix} \right\} \subset R^3$ as shown in the figure 12.4 is not a subspace of R^3 because the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in W.

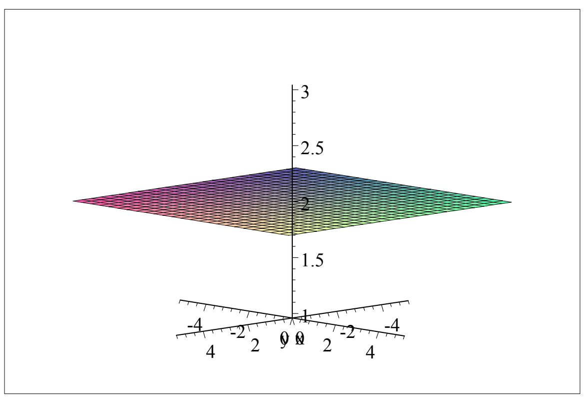


Figure 12.4

Example 12.5: Recall that the column space of a matrix is the span of the columns of the matrix A, it is a subspace because Span of any subset forms a subspace. The column space of a matrix A is denoted by ColA.

Suppose A= $\begin{bmatrix} 1 & 2 & 3 & 4 & 8 \\ 1 & 1 & 1 & 2 & 4 \\ -1 & 1 & 3 & 0 & 4 \end{bmatrix}$.

If we want to determine whether
a vector b∈R^3 is in ColA,
we should check if b is a linear combination of the columns of A, which is equivalent to saying that Ax=b has a solution.

Let us take

b= $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

To check if b is in the ColA, all that we have to see is if the augmented matrix [A|b] corresponds to a consistent system.

Look at

[A|b]= $\begin{bmatrix} 1 & 2 & 3 & 4 & 8 & 2 \\ 1 & 1 & 1 & 2 & 4 & 1 \\ -1 & 1 & 3 & 0 & 4 & 1 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

Since, we do not have any row of the form

$0 \quad 0 \quad 0 \quad 0 \quad 0$ non zero number

the system is consistent and b∈ColA.

Moreover:

Since each row in A has a pivot position, any vector in R^3 is in ColA by the INVERTIBLE MATRIX THEOREM. for A= $\begin{bmatrix} 1 & 2 & 3 & 4 & 8 \\ 1 & 1 & 1 & 2 & 4 \\ -1 & 1 & 3 & 0 & 4 \end{bmatrix}$.

Example 12.6:

To check if b= $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$ is in ColA, where A= $\begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 1 & 2 & 1 & 0 & -3 \\ -1 & 1 & 1 & -4 & -2 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix}$

Look at

[A|b]= $\begin{bmatrix} 1 & 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 1 & 0 & -3 & -1 \\ -1 & 1 & 1 & -4 & -2 & 1 \\ 1 & 1 & 1 & 2 & 0 & 2 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

since the last row of this matrix has a nonzero entry in the pivot column only, the system is inconsistent and b can not be in ColA.

Recall that a basis of a H subspace is a linealry independent subset of the subspace that spans. For example, we take R^2 as a subspace of R^2,

then $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of R^2. You may check that

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is another basis of R^2.

Finding a basis of NulA.

Recall that $x \in R^n$ is in the null space of $A_{m\times n}$, if Ax=0

Let us look at the null space of the following matrix.

A= $\begin{bmatrix} 1 & 2 & 3 & 4 & 8 \\ 1 & 1 & 1 & 2 & 4 \\ -1 & 1 & 3 & 0 & 4 \end{bmatrix}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in R^5$$

iff

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 8 \\ 1 & 1 & 1 & 2 & 4 \\ -1 & 1 & 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Look at the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 8 & 0 \\ 1 & 1 & 1 & 2 & 4 & 0 \\ -1 & 1 & 3 & 0 & 4 & 0 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

x_3 and x_5 are free variables and we have

$x_1 - x_3 = 0$

$x_2 - 2x_3 - 4x_5 = 0$

$x_4 = 0$

therefore

$x_1 = x_3$

$x_2 = 2x_3 + 4x_5$

$x_3 = x_3$

$x_4 = 0$

$x_5 = x_5$

$$\text{or } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

therefore, each x in NulA can be expressed as a linear combination of

$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Since, both these vectors are linearly independent,

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of NulA.

To find a basis of ColA where A= $\begin{bmatrix} 1 & -1 & 4 & 1 \\ 1 & -1 & 4 & 1 \\ -1 & -5 & 10 & 1 \end{bmatrix}$

By theorem 13 (in section 2.8 in the third edition or 2.8 in the second updated edition) the pivot columns of A form a basis of ColA.

First, we row reduce A and then identify the columns corresponding to the pivot columns in the original matrix A.

A= $\begin{bmatrix} 1 & -1 & 4 & 1 \\ 1 & -1 & 4 & 1 \\ -1 & -5 & 10 & 1 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & \frac{5}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{7}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of ColA is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -5 \end{bmatrix} \right\}$

Exercise 38 on page 175 in the third edition.

Given that $A = \begin{bmatrix} 5 & 2 & 0 & -8 & -8 \\ 4 & 1 & 2 & -8 & -9 \\ 5 & 1 & 3 & 5 & 19 \\ -8 & -5 & 6 & 8 & 5 \end{bmatrix}$ to find basis of NulA and ColA.

For NulA:

$\begin{bmatrix} 5 & 2 & 0 & -8 & -8 & 0 \\ 4 & 1 & 2 & -8 & -9 & 0 \\ 5 & 1 & 3 & 5 & 19 & 0 \\ -8 & -5 & 6 & 8 & 5 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 60 & 122 & 0 \\ 0 & 1 & 0 & -154 & -309 & 0 \\ 0 & 0 & 1 & -47 & -94 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

therefore a basis of NulA is $\left\{ \begin{bmatrix} -60 \\ 154 \\ 47 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -122 \\ 309 \\ 94 \\ 0 \\ 1 \end{bmatrix} \right\}$

For ColA:

note that the first three columns of A are the pivot columns.

Therefore a basis is $\left\{ \begin{bmatrix} 5 \\ 4 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 6 \end{bmatrix} \right\}$.