Lesson 10:

This lesson should help you work on section 1.9 in the third edition or Section 1.8 in the second updated edition.

In the last section, we saw many examples of linear transformations with definition
x→Ax where A is a matrix of appropriate size.

In this section, we shall see that for any linear transformation T:$R^n \to R^m$**, we can find a**
matrix A$_{m\times n}$ **such that T(x)=Ax.**

The construction of A is based on the proof of the theorem 10 in section 1.9 in the third edition and Section 1.8 in the second updated edition.

Here is an example:

Let T: $R^3 \to R^2$ **be a linear transformation.**

any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Therefore

T$\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = x_1 T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + x_2 T \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + x_3 T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

For example,

Let T: $R^3 \to R^2$ **be defined by** $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 + 7x_3 \\ x_1 + x_2 - 3x_3 \end{bmatrix}$

Note that $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mathbf{x}_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mathbf{x}_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{T}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}_1 \mathbf{T}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mathbf{x}_2 \mathbf{T}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mathbf{x}_3 \mathbf{T}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\to \mathbf{T}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mathbf{x}_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \mathbf{x}_3 \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

$$\to \mathbf{T}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 7 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Therefore if A= $\begin{bmatrix} 2 & -1 & 7 \\ 1 & 1 & -3 \end{bmatrix}$

T(x)=Ax

We call such a matrix the standard matrix of the linear transformation T.

Notation:

In R^3, we use the noations $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_3$

Note that these vectors are the columns of the 3x3 Identity matrix. Use similar notations
for R^n for any n.

As another example, we may look at the exercise 2 on the page 90 of the third edition,
which gives us T: $R^3 \to R^2$, defined by T(e_1)=(1,3), T(e_2)=$(4,-7)$, T(e_3)=$(-5,4)$. To find the standard matrix of T.

Just as above, the standard matrix is obtained by arranging the images of

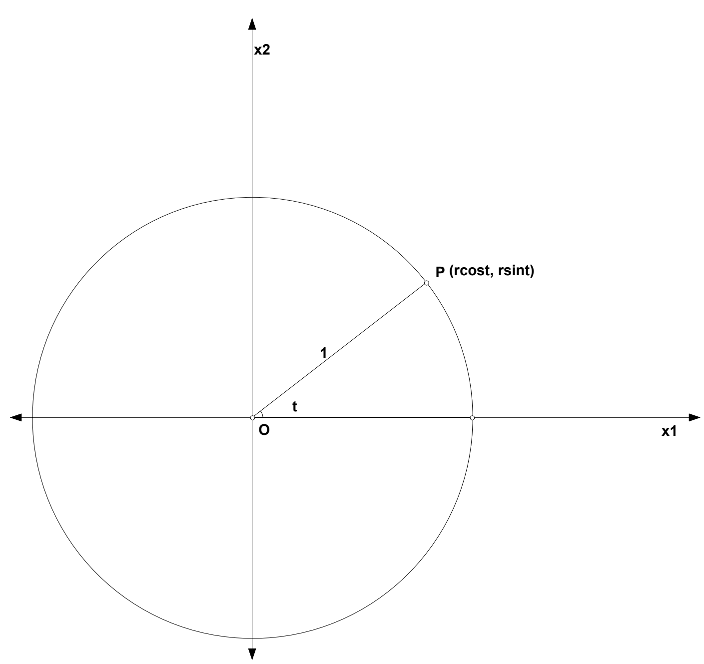
e_1, e_2, e_3 as the columns of the matrix in order.

i.e. the standard matrix $= \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$

Please read Theorem 10 and understand the proof and please make sure that you know all the geometric linear transformations of R^2 that are illustrated in this section.

Note that to define a linear transformation, T:$R^n \to R^m$, it is enough to define it on the column vectors of $I_{n \times n}$ the nxn identity matrix.

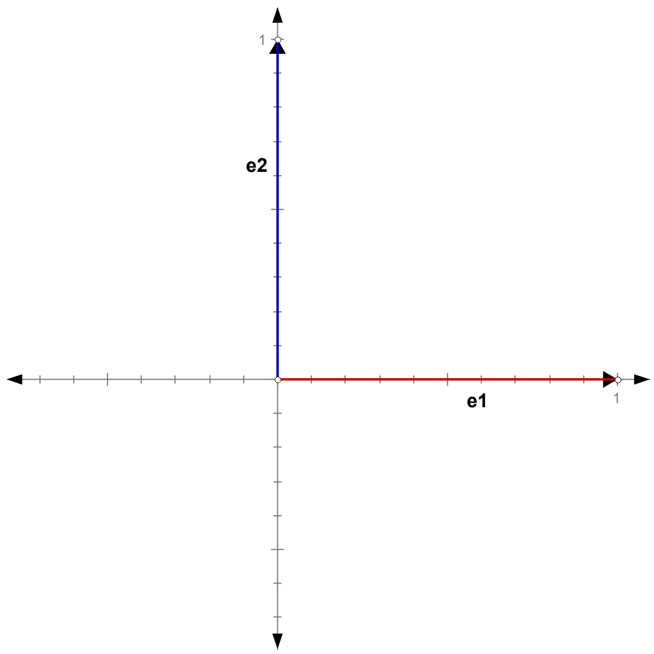
Before proceeding ahead, let us recall from Unit Circle Trigonometry, that for a circle of unit radius and center O, the coordinates of a point P on the corcumference can be written in terms of the angle t that OP subtends with the x_1-axis.



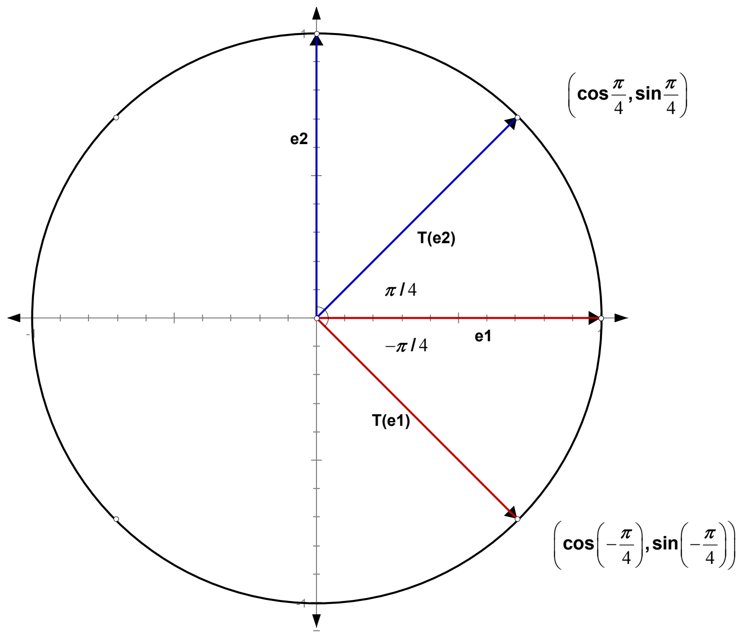
Exercise 4 on page 90 in the third edition:

To obtain the standard matrix of the linear transformation T:$R^2 \rightarrow R^2$ **that rotates the points**
in the plane about the origin through $-\frac{\pi}{4}$**.**

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **,** $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$**,**



to obtain the standard matrix, it is enough to obtain
T(e_1**) and T(**e_2**)**



Therefore

$$\mathbf{T}(e_1)=\begin{bmatrix} \cos\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) \end{bmatrix} \rightarrow \mathbf{T}(e_1)=\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{T}(e_2)=\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) \end{bmatrix} \rightarrow \mathbf{T}(e_2)=\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Therefore the standard matrix of this linear transformation is

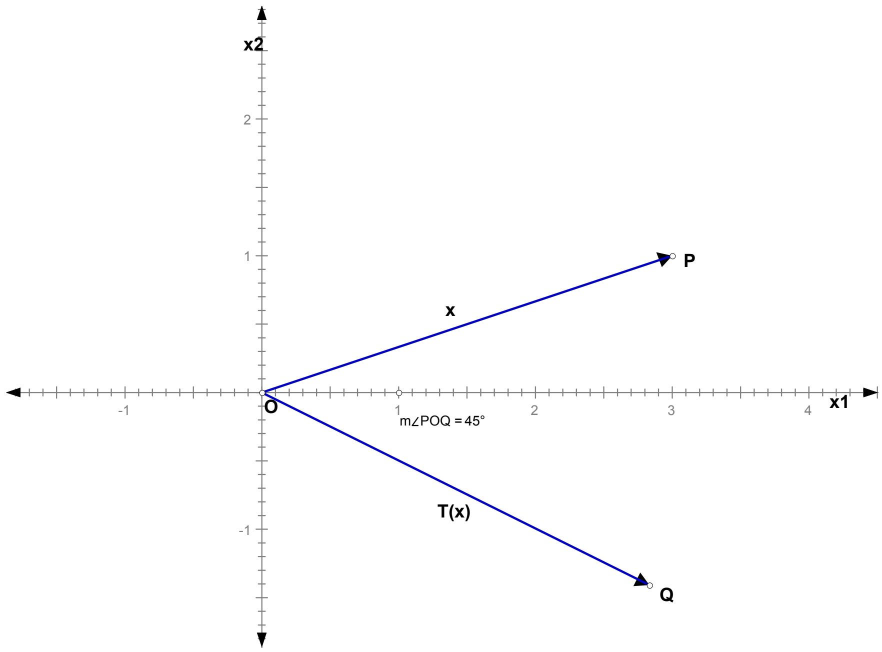
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The exercise is done, but just to see how this works, take any vector in R^2 say

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\mathbf{T(x)}=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$$

Note that $\begin{bmatrix} 2\sqrt{2} \\ -\sqrt{2} \end{bmatrix} \cong \begin{bmatrix} 2.8284 \\ -1.4142 \end{bmatrix}$



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6. To write the standard matrix of T:$R^2 \to R^2$

given that T(e_1**)=**e_1 **T(**e_2**)=**$e_2 + 3e_1$**.**

T(e_1**)=**e_1 $\to$**T(**e_1**)=**$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

T(e_2**)=**$e_2 + 3e_1$ $\to$**T(**e_2**)=**$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\to$**T(**e_2**)=**$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

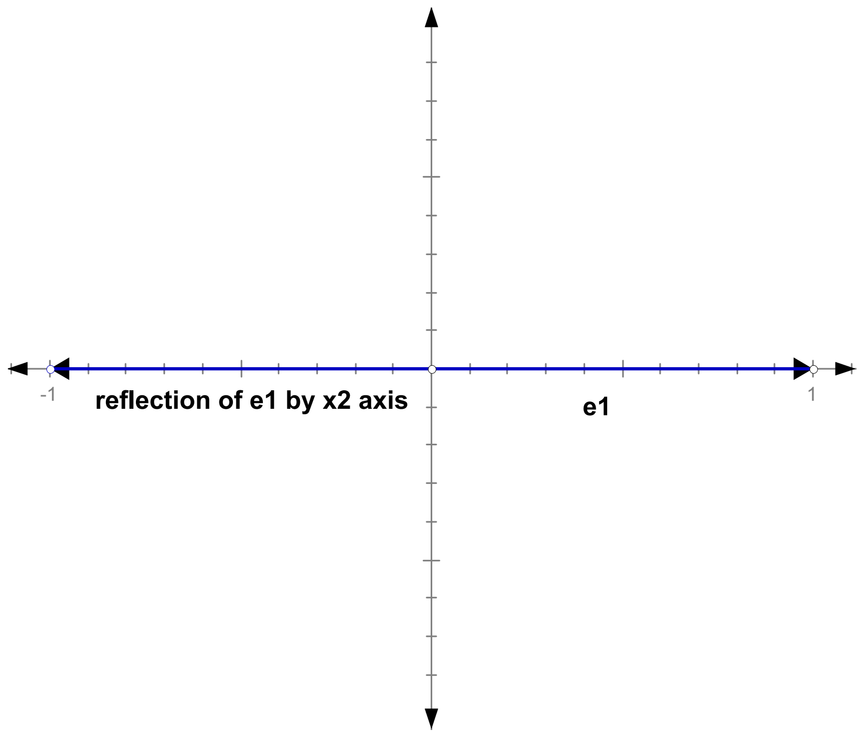
Therefore the standard matrix is A=$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

10.
Given a linear transformation T:$R^2 \to R^2$ **which first reflects points through the vertical**

x_2 -axis and then rotates the points $\frac{\pi}{2}$ radians. To find the standard matrix.

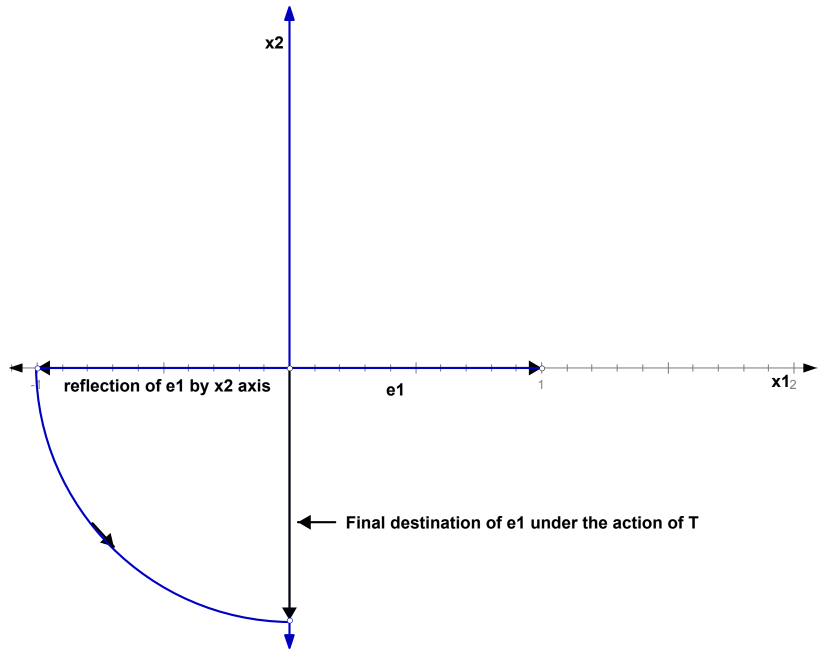
First, let us look at T(e_1)

First, let us look at the reflection of e_1 by the x_2-axis.



Note that the reflection maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **to** $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Now the rotation by $\frac{\pi}{2}$ will map it to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ **as shown below**



Note that reflection of e_2 by the x_2-axis is e_2 itself, when rotated by $\frac{\pi}{2}$ counter clock wise

it should land on $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Therefore

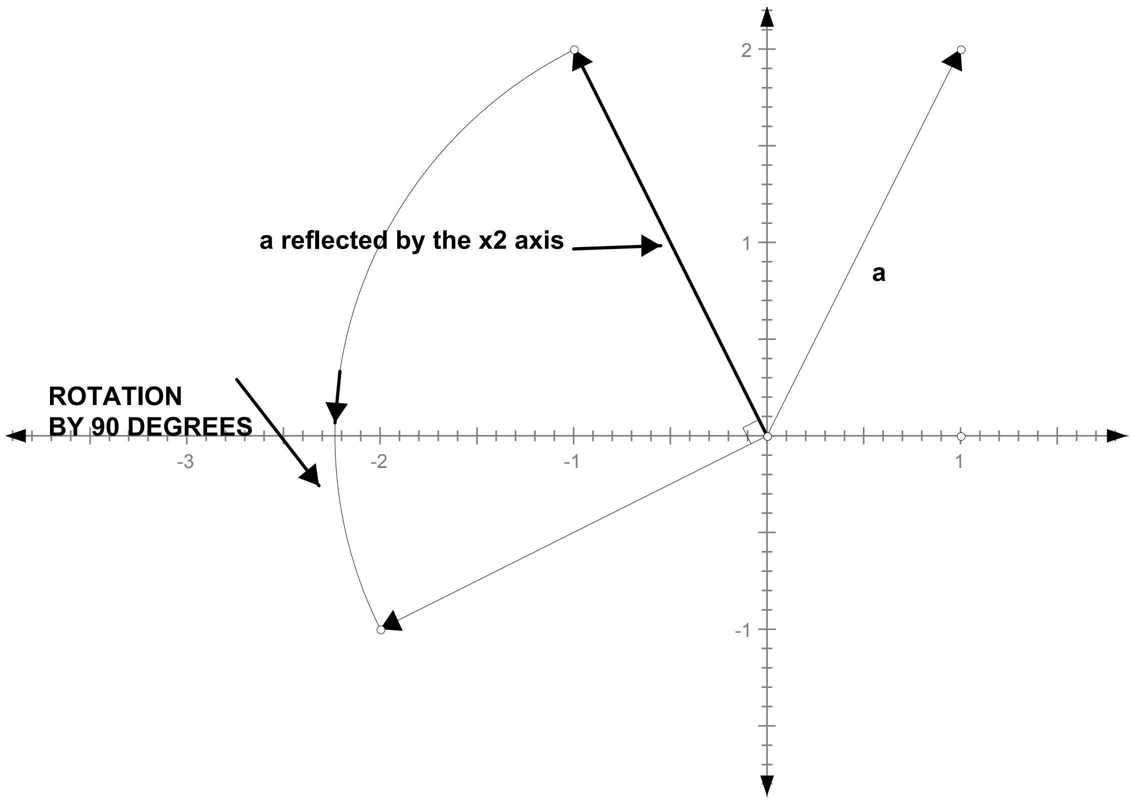
T$(e_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\qquad T(e_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

and the standard matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

The question is answered, still let us verify this on a=$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in R^2$**.**

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **when reflected by the x_2 axis goes to** $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **when rotated counter clock wise**

by $\frac{\pi}{2}$ it is mapped to $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ **as shown below.**



and also

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

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18. in section 1.9 the third edition

Given that

T$(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

Remember, to obtain the matrix of this linear transformation, all that we have to do is find

$T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **and** $T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ **and show that the above mapping can be obtained by multiplication by**

A$= \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$.

T $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

T $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$

A$= \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

Ax$= \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 + 2x_2 \\ x_1 - 4x_2 \\ 0 \\ x_2 \end{bmatrix}$

Therefore T(x)=Ax and T:$R^2 \to R^4$ **is a linear transformation.**

Exercise #22 section 1.9 Third Edition:

Given a linear transformation T:$R^2 \to R^3$ **such that**
$T(x_1,x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$
To find x such that $T(x_1,x_2) = (-1,4,9)$

Solution:
we may re write the map as

T $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

To find x$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **such that T(x)**$= \begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix}$

is the same as to find x= $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **such that**

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix}$$

Look at the augmented matrix

$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 4 \\ 3 & -2 & 9 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

Therefore the answer is x= $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Check:

T$(5,3)=(5-2\times 3,-5+3\times 3,3\times 5-2\times 3)=\ (-1,4,9)$

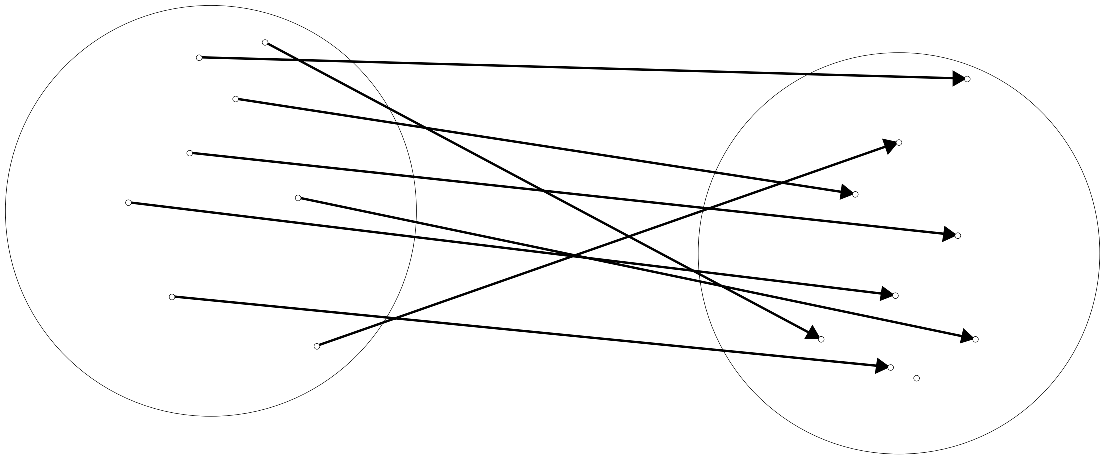
Note:

From this point on we shall put a strong emphasis on writing proofs. I shall write some proofs in the lessons. The best way to learn writing proofs is to read the text has very nicely written for you.

Before you read the rest of the lesson, please read the definitions of an onto mapping and
a one to one mapping on the page 87 in the third edition and page ..... in the second updated edition.

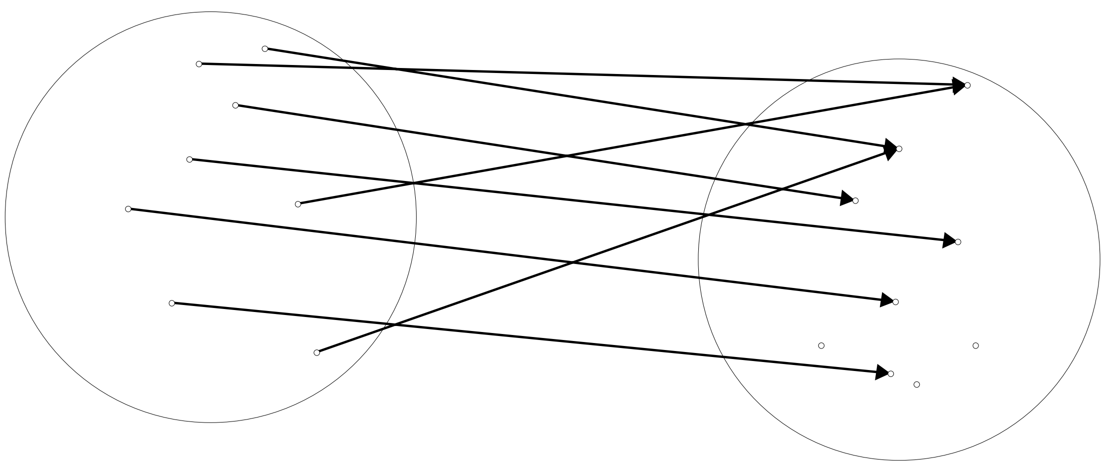
Recall that a one-one transformation assigns to each element in the domain a unique element in the range.

For example:



Is a one-one map.

But



is NOT one-one.

Algebraically: T:$R^n \to R^m$ **is one-one if T(u)=T(v)**$\Rightarrow u = v$**.**

Note that the theorems 10 and 11 (please make sure that you read and understand the proofs) in the section 1.9 the third edition or 1.8 in the second updated edition, state
that
A linear transformation T:$R^n \to R^m$ **with matrix A is one-one**
iff
T(x)=0$\Rightarrow$**x=0**
or
iff

The columns of A are linearly independent.

To see an illustration, consider

$T : R^3 \to R^3$ **defined by T** $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -4 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -4 \\ 2 & 1 & 5 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Note that the columns are linearly dependent therefore the linear transformation is not one-one.

Furthermore, note that

T $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -4 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

as well as

T $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -4 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Showing that there are more than one element with $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **as the image.**

Recall that a linear transformation T:$R^n \to R^m$ **is onto iff for each b in** R^m **we can find x in** R^n **such that T(x)=b.**
In view of the theorem 12 in section 1.9 Third Edition or in the section 1.8 in the second updated edition
If A is the standard matrix of T then T is onto iff the columns of A span R^m **or in other words in each row in A has a pivot position.**

From the text:

26 on page 91 in the third edition:

To check if T:$R^3 \rightarrow R^2$ given by

T$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, **T**$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, **T**$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$**is**

a) one-one

b) onto

First note that the standard matrix A=$\begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$

Since the columns of this matrix can not be linearly independent (why?), therefore T is not one one.

additional practice, find u and v in R^3 such that T(u)=T(u)=$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

A=$\begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

each row has a pivot position therefore T is onto.

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38 on page 91 in the third edition:

To check if T:$R^4 \rightarrow R^4$ with standard matrix A=$\begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & -6 & -2 & 5 \end{bmatrix}$ **is one to one.**

Since

$\begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & -6 & -2 & 5 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

columns are not linearly independent, the transformation is not one-one.

40 on page 91 in the third edition:

To check if T:$R^5 \to R^5$ **with standard matrix A=** $\begin{bmatrix} 9 & 13 & 5 & 6 & -1 \\ 14 & 15 & -7 & -6 & 4 \\ -8 & -9 & 12 & -5 & -9 \\ -5 & -6 & -8 & 9 & 8 \\ 13 & 14 & 15 & 2 & 11 \end{bmatrix}$ **is onto.**

$\begin{bmatrix} 9 & 13 & 5 & 6 & -1 \\ 14 & 15 & -7 & -6 & 4 \\ -8 & -9 & 12 & -5 & -9 \\ -5 & -6 & -8 & 9 & 8 \\ 13 & 14 & 15 & 2 & 11 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Since the last row does not have any pivot position, it is not Onto.

Additional exercise: Find b in R^5 such that there is no x in R^5 with T(x)=b