Lesson 4:

First read the definitions in the section 1.3 and then check if you can recall the follwoing:

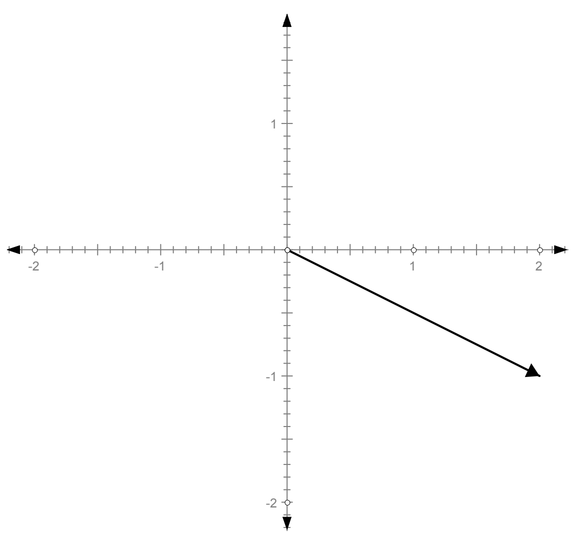
Vectors in R^2:

Terminology:

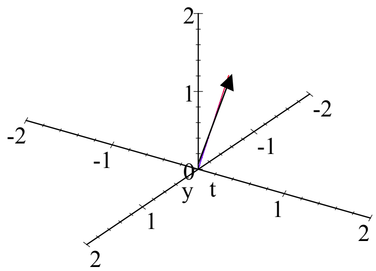
A matrix with only one column is called a column vector or simply a vector.

We shall regard the two dimensional plane as R^2, **the set of all points in the coordinate plane**.

A vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **will be represented by**



While in R^3, **the vector** $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ **can be displayed by**



Also recall that

$$\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 7 \\ 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 6 \\ 4 \\ 3 \end{bmatrix}$$ **in** R^4.

and

$$2\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \\ 10 \end{bmatrix}$$ **in** R^5.

Suppose that we are in R^n, **where n is a positive integer and that** $v_1, v_2, ..., v_k$ **are vectors in** R^n.

If $_1,\ _2, ...,\ _k$ **are real numbers then** $_1v_1 \quad _2v_2 \quad ..... \quad _kv_k$ **is a linear combination of** $v_1, v_2, ..., v_k$ **in** R^n.

For example:

Consider $v_1 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, v_2 \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, v_3 \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, v_4 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ **in** R^5.

Then $2\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \quad 3\begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \quad .5\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ **is a linear combination of**

v_1, v_2, v_3, v_4 **in** R^5.

or note that

$$\begin{bmatrix} 4.5 \\ 1.0 \\ 7 \\ 4.0 \\ 8.5 \end{bmatrix} \quad \mathbf{2}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{3}\begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \quad \mathbf{.5}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

we say that $\begin{bmatrix} 4.5 \\ 1.0 \\ 7 \\ 4.0 \\ 8.5 \end{bmatrix}$ **is a linear combination of** v_1, v_2, v_3, v_4 **with the weights**

$2, \; 3, 1, .5$ **respectively**.

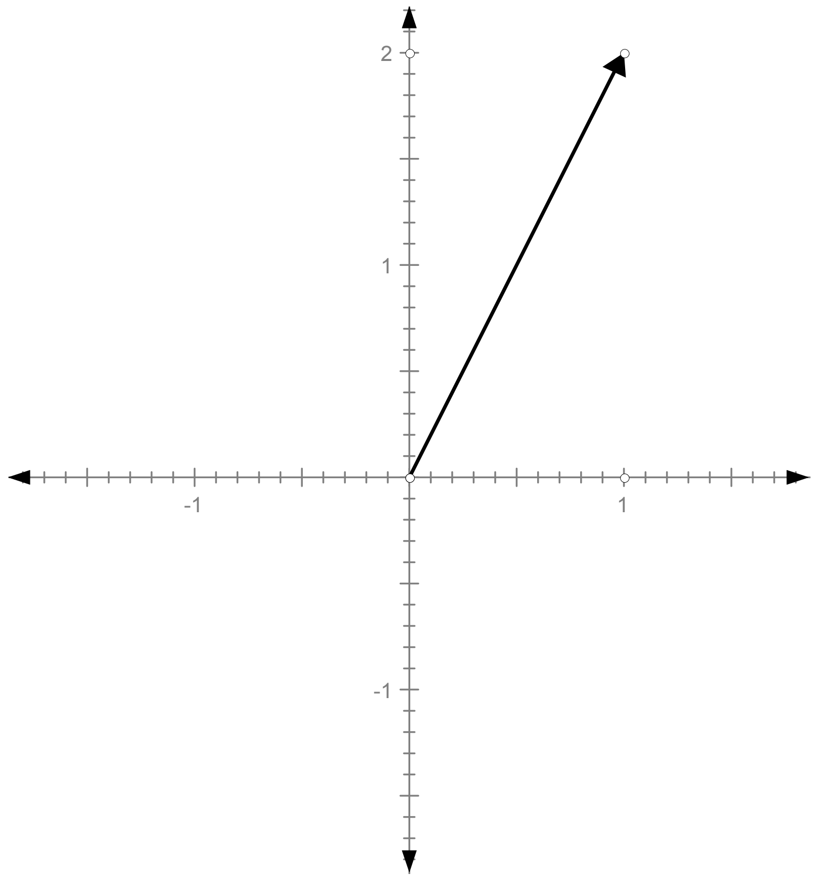
If $v_1, v_2, ..., v_k$ **are vectors in** R^n **then Span** $\{v_1, v_2, ..., v_k\}$ **is the subset containing all possible linear combinations of**

$v_1, v_2, ..., v_k$ **in** R^n.

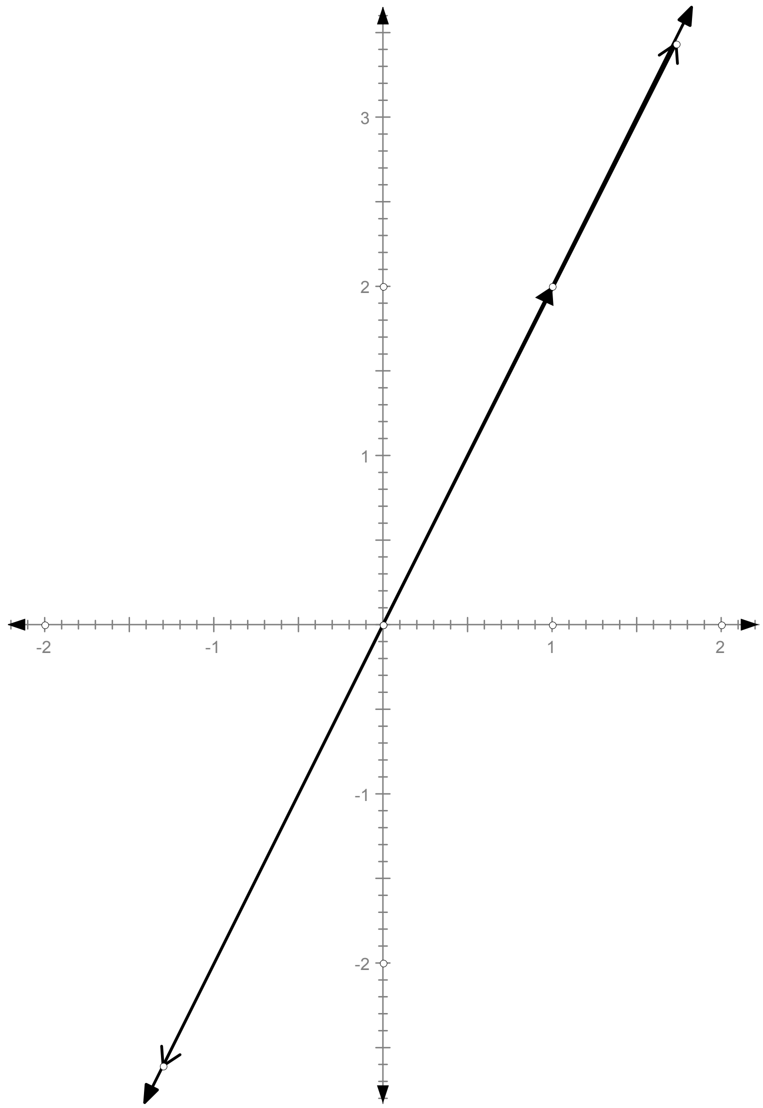
If you do not understand the following statements, please post your questions on the web site.

Example 1.

Consider $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **in** R^2.

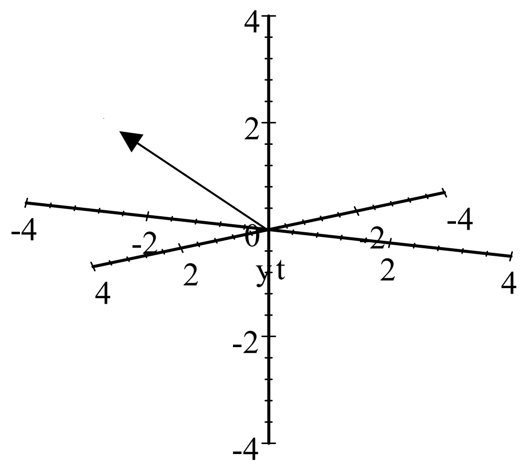


Span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ **is displayed below**.

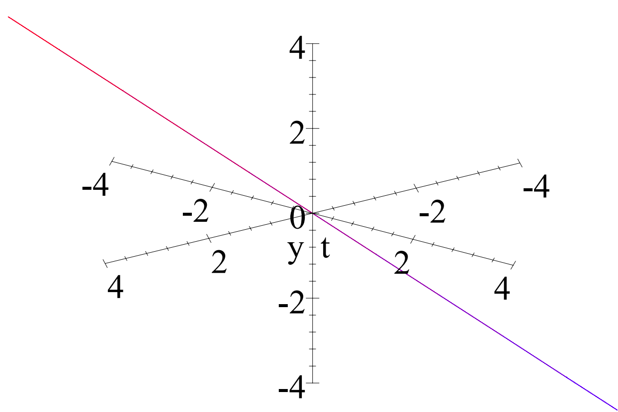


Example 2.

Consider $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ **in** R^3.



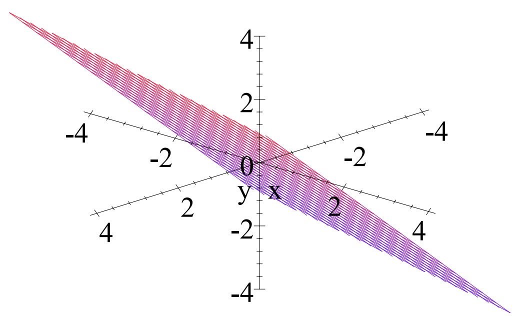
span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$ **is the set of all the vectors on the line**.



Example 3:

Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

is



Note that we can use vector equations to represent a system of linear equations.

For example, let us take

the exercise #10 in section 1.3 of the third edition

$4x_1 \;\; x_2 \;\; 3x_3 \quad 9$
$x_1 \;\; 7x_2 \;\; 2x_3 \quad 2$
$8x_1 \;\; 6x_2 \;\; 5x_3 \quad 15$

We have to write this system as a vector equation.

(Note that if you do have the second updated edition, you should still be able to follow.)

You can easily note that a vector notation is

$$\mathbf{x}_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \mathbf{x}_2 \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix} \mathbf{x}_3 \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$$

Exercise 12 in the third edition: (Note that if you do have the second updated edition, you should still be able to follow.)

We have to determine if b $\begin{bmatrix} 5 \\ 11 \\ 7 \end{bmatrix}$ **is a linear combintaion of a**$_1$ $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$,

a$_2$ $\begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$, **and a**$_3$ $\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$.

In order for b to be a linear combination of a$_1$,**a**$_2$,**a**$_3$ **we have to see if it is possible to find coefficients** x_1, x_2, **and** x_3
such that

$x_1\mathbf{a}_1 \;\; x_2\mathbf{a}_2 \;\; x_3\mathbf{a}_3 \;\; \mathbf{b}$

or

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \; x_2 \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} \; x_3 \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \; \begin{bmatrix} 5 \\ 11 \\ 7 \end{bmatrix}$$

or in other words
we have to check if the augnmented matrix,

$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 2 & 5 & 0 & 11 \\ 2 & 5 & 8 & 7 \end{bmatrix}$ **corresponds to a consistent system of linear equations.**

For the above augmented matrix, note

$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 2 & 5 & 0 & 11 \\ 2 & 5 & 8 & 7 \end{bmatrix}$, **row echelon form:** $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Note that in the last row, the only non zero entry is in the entry corresponding to the augmented column, therefore the system is inconsistent, and no such

x_1, x_2, **and** x_3 **will be available. (recall theorem 2 in section 1.2.)**

Therefore b is not a linear combination of a$_1$, **a**$_2$, **a**$_3$.

Exercise 14 on page 38 in the third edition.

(Note that if you do have the second updated edition, you should still be able to follow.)

We have to determine if b $\begin{bmatrix} 11 \\ 5 \\ 9 \end{bmatrix}$ **is a linear combination of the columns of the matrix A** $\begin{bmatrix} 1 & 2 & 6 \\ 0 & 3 & 7 \\ 1 & 2 & 5 \end{bmatrix}$.

Provide reason (....I am leaving these for you to fill in...) as to why, is it equivalent to checking if the augmented matrix

$\begin{bmatrix} A & b \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 6 & 11 \\ 0 & 3 & 7 & 5 \\ 1 & 2 & 5 & 9 \end{bmatrix}$

corresponds to a consistent system.
Note that

$$\begin{bmatrix} 1 & 2 & 6 & 11 \\ 0 & 3 & 7 & 5 \\ 1 & 2 & 5 & 9 \end{bmatrix}, \textbf{ row echelon form: } \begin{bmatrix} 1 & 0 & 0 & \frac{245}{33} \\ 0 & 1 & 0 & \frac{41}{33} \\ 0 & 0 & 1 & \frac{2}{11} \end{bmatrix}$$

this indicates that b is a linear combination of the columns of the matrix A, moreover you may note that

$$\frac{245}{33}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{41}{33}\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \frac{2}{11}\begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix} \begin{bmatrix} 11 \\ 5 \\ 9 \end{bmatrix}.$$