Before reading this lesson, please read
the definition of orthogonally diagonizable in section 7.1
the Spectral Theorem for Symmetric Matrices in section 7.1
the definition of Quadratic form in the section 7.2
The Principal Axes Theorem in the section 7.2

Recall from your readings that

If we have a symmetric matrix
i.e. $A = A^T$
Then A is always diagonalizable.

In this lesson, we shall see examples of using the orthogonal diagonalization to transform a quadratic form into a quadratic form without any cross product term.

For Example

Look at the Quadratic form

$$3x_1^2 - 4x_1x_2 + 6x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Since, we are in two dimensions for the sake of simplicity let us write

$3x_1^2 - 4x_1x_2 + 6x_2^2 = 3x^2 - 4xy + 6y^2$

that is use $x = x_1$, $y = x_2$

Look at the matrix of the quadratic form above which is

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

Find the Eigen Values

$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$, eigenvalues: $2, 7$

Then find the eigen vectors

Eigen vectors

For the eigen value 2:

$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ an eigen vector for 2

Normalize this vector and get

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For the eigen value 7

$\begin{bmatrix} -4 & -2 & 0 \\ -2 & -1 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ an eigen vector for 7

Normalize this vector and get

$$\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Normalize this vector and get

$$\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Use the transformation

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}$$

or

$$\begin{bmatrix} \frac{2}{5}\sqrt{5}\,u - \frac{1}{5}\sqrt{5}\,v \\ \frac{1}{5}\sqrt{5}\,u + \frac{2}{5}\sqrt{5}\,v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Apply the transformation

$3x^2 - 4xy + 6y^2$

$= 3\left(\frac{2}{5}\sqrt{5}\,u - \frac{1}{5}\sqrt{5}\,v\right)^2 - 4\left(\frac{2}{5}\sqrt{5}\,u - \frac{1}{5}\sqrt{5}\,v\right)\left(\frac{1}{5}\sqrt{5}\,u + \frac{2}{5}\sqrt{5}\,v\right)$

$+6\left(\frac{1}{5}\sqrt{5}\,u + \frac{2}{5}\sqrt{5}\,v\right)^2$

$= 2u^2 + 7v^2$

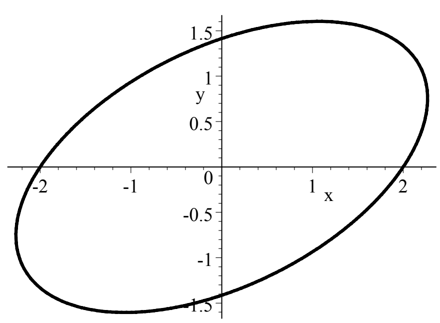
Note that

$$2u^2 + 7v^2 = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Let note a geometrical application of this procedure:

Note that a graph of the equation

$3x^2 - 4xy + 6y^2 = 12$ is the ellipse



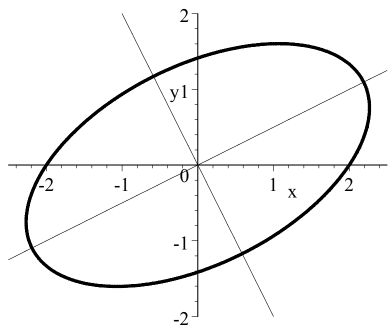
If we apply the above transformation that is take the axes along the vectors

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

that is along the lines with equations $2y = x$ and $y = -2x$

the equation changes to
$2u^2 + 7v^2 = 12$

and the graph of the ellipse is much easier to obtain.



Please work on the assigned exercises in the sections 7.1 and 7.2
and post your questions in the discussion area.