Lesson 3:

Please make sure that you are proficient with the following terminologies.

1. **Leading entry**
2. **Echelon form**
3. **Reduced Echelon Form**

If not, please read the lesson 2 and also the pages 14 and 15 of the text book to review the above.

Now we are going to understand the meanings of the terms
pivot position
and
pivot column
as explained on the pages 16 and 17 of the text book.
Consider the matrix

$$\begin{bmatrix} 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that it is in the row reduced echelon form.

Noattion: Position (i,j) means the position corresponding to the row#i and column #j. For example: Position (2,4) corresponds to row #2 and column #4.

Since, the position (1,1) i.e. row 1& column 1 corresponds to a leading entry in the row reduced echelon form, we say that the position (1,1) is a pivot position and the column containing this pivot position namely the column 1 is a pivot column.

Likewise note that the position (2,3) i.e. row 2& column 3 corresponds to a leading entry in the row reduced echelon form, we say that the position (2,3) is a pivot position and the column containing this pivot position namely the column 3 is a pivot column.

Likewise note that the position (3,5) i.e. row 3 & column 5 corresponds to a leading entry in the row reduced echelon form, we say that the position (3,5) is a pivot position and the column containing this pivot position namely the

column 5 is a pivot column.

The follwoing summarizes the above discussion.

Pivot position   pivot position

$$\begin{bmatrix} 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot position

Pivot col.   Pivot column   Pivot Column

These two terms will help us understand many definitions and procedures. For example, let us consider the matrix

$$\textbf{A} \begin{bmatrix} 2 & 1 & 3 & 1 & 1 \\ 1 & 5 & 1 & 2 & 3 \\ 4 & 11 & 5 & 5 & 5 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

To reduce this matrix to the row reduced echelon form, the following is the most methodical (not the only though) procedure.

Step 1. Use the row operations to transform the column 1 into a pivot column. You may call this procedure

Pivoting the first column

First we need 1 in the position (1,1)

This may be done by $r_1 \quad r_4$**, note that interchanging with row would have done the same thing but the entries in row 4 easier to handle.**

Therefore we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 5 & 1 & 2 & 3 \\ 4 & 11 & 5 & 5 & 5 \\ 2 & 1 & 3 & 1 & 1 \end{bmatrix}$$

Now we need, in the first column, all the entries except the top, to be zeros. This may be done by

$r_2 \quad r_1, \quad r_3 \quad 4r_1, \quad r_4 \quad 2r_1$

producing

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 4 & 0 & 3 & 1 \\ 0 & 7 & 1 & 9 & 3 \\ 0 & 1 & 1 & 3 & 5 \end{bmatrix}$$

Step2:

Pivot the second column, making sure that the first column is not changed. To obtain 1 in the postition (2,2) simply

$r_2 \qquad r_4$ **to obtain**

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 5 \\ 0 & 7 & 1 & 9 & 3 \\ 0 & 4 & 0 & 3 & 1 \end{bmatrix}$$

then perform $1 \; r_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 5 \\ 0 & 7 & 1 & 9 & 3 \\ 0 & 4 & 0 & 3 & 1 \end{bmatrix}$$

Now to transform the second column so that all the entries except the second become zero, may do, with the use of the second row

$r_1 \quad r_2, \quad r_3 \quad 7r_2, \quad r_4 \quad 4r_2$

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & 1 & 3 & 5 \\ 0 & 0 & 8 & 30 & 38 \\ 0 & 0 & 4 & 15 & 19 \end{bmatrix}$$

Step3:

Pivot the third column, making sure that the first and the secondcolumn are not changed.

To obtain 1 in the position (3,3) , we may perform $\frac{1}{8} r_2$ **to obtain**

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & 1 & 3 & 5 \\ 0 & 0 & 1 & \frac{15}{4} & \frac{19}{4} \\ 0 & 0 & 4 & 15 & 19 \end{bmatrix}$$

To transform all the entries other than the pivot position in the column 3 to zero, without disturbing the first and the second columns,

of course the third row is a good one to use

one may

$r_1 \quad 2r_3$, $r_2 \quad r_3, r_4 \quad 4r_3$

to obtain

$$\begin{bmatrix} 1 & 0 & 0 & \frac{11}{2} & \frac{13}{2} \\ 0 & 1 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{15}{4} & \frac{19}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the reduced echelon form of the matrix A.

More terminlogies:

Suppose that

$$\mathbf{A} \begin{bmatrix} 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is the reduced echelon form of a systems of linear equations in the variables x_1, x_2, x_3, x_4 **in the following manner**

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_3, x_4 correspond to the three pivot columns and are called basic variables, the other variable x_2 is

We may use this form to write the solution of a system of linear equations.

For example, in the above situation,

$$\begin{array}{l} x_1 \quad 3x_2 \quad 5 \\ \qquad x_3 \quad 3 \\ \qquad x_4 \quad 0 \end{array}$$

or

$$\begin{array}{l} x_1 \quad 5 \quad 3x_2 \\ x_3 \quad 3 \\ x_4 \quad 0 \end{array}$$

Assigning different values to x_2 (the free variable in this case) we can obtain different solutions of the system of linear equations under consideration.
Of course, when we do not have any free variable in the row reduced echelon form of the augmented matrix of a consistent system, the solution will be unique. An inconsistent system will arise, when in the row reduced form, the augmented column is a pivot column, i.e. there is a row in which all the entries except the one in the augmented column are zeros.

Example of an inconsistent system:

Consider,
$x_1 \quad 2x_2 \quad x_3 \quad 1$
$3x_1 \quad 6x_2 \quad 3x_3 \quad 20$
$x_1 \quad x_2 \quad x_3 \quad 4$

The augmented matrix is

$$\begin{array}{ccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 3 & -6 & 3 & 20 \\ 1 & 1 & 1 & 4 \end{array}\right]$$

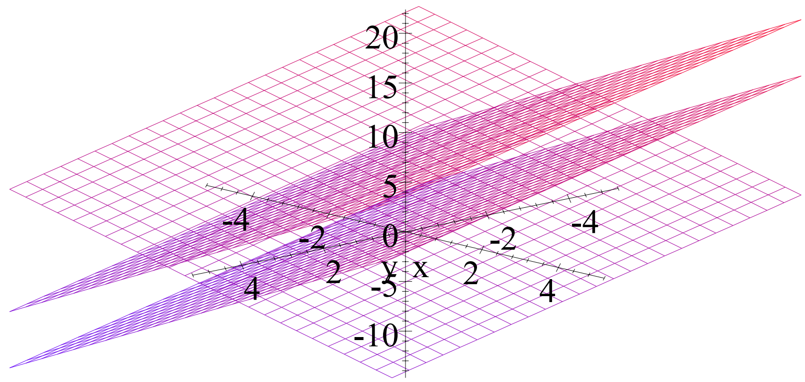
with the row reduced echelon form

$$\begin{array}{ccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Note that the last row will correspond to
0 1(bad) , which implies that the system does not have a solution.

Graphically, the three planes will look like



which do not share any common point.

Now you should be ready to work on the exercises in the section 1.2.

Please ask the problems that give you difficulty.

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