Lesson 29:

This lesson will mainly focus on using the Gram Schmidt process to vector spaces other than R^n.

Before reading the lesson make sure that you have read the definition of inner product in the section 6.7 of the text. Also make sure to read the about length, distance, and orthogonality.

Recall that

$<f,g> = \int_{-1}^{1} f(x)g(x)dx$ defines an inner product on the vector space $C[0,1]$

Note that P_2 the vector space of polynomials of degree less than or equal to 2 is a subspace of C[0,1].

B=$\{1,x,x^2\}$ is a basis of P_2.
Let us check if B is an orthogonal basis.

$<1,x> = \int_{-1}^{1} 1x dx = 0$ therefore 1 and x are orthogonal

$<x,x^2>=\int_{-1}^{1} xx^2 dx = 0$ therefore x and x^2 are orthogonal

$<1,x^2>=\int_{-1}^{1} 1x^2 dx = \frac{2}{3}$ therefore 1 and x^2 are not orthogonal.

Let us develop an orthogonal basis of P_2 along the line of the Gram Schmidt process.

We know that 1 and x are orthogonal.
We shall take the comonent of x^2 that is orthogonal to the Span{1,x^2} in the following manner

$$x^2 - \frac{<1,x^2>}{<1,1>}1 - \frac{<x,x^2>}{<x,x>}x$$

$$= x^2 - \frac{\int_{-1}^{1} 1x^2 dx}{\int_{-1}^{1} 1 \times 1 dx} 1 - 0$$

$$= x^2 - \frac{(2/3)}{2}$$

$$= x^2 - \frac{1}{3}$$

Note that

$$\int_{-1}^{1} 1\left(x^2 - \frac{1}{3}\right) dx = 0$$

$$\int_{-1}^{1} x\left(x^2 - \frac{1}{3}\right) dx = 0$$

Therefore $\left\{1, x, x^2 - \frac{1}{3}\right\}$ is an orthogonal basis of P_2.

For an orthonormal basis, we just need to normalize these vectors, that is multiply them by the reciprocal of their norm.

Note that

$$< 1, 1 >= \int_{-1}^{1} 1 \times 1 dx = 2$$

therefore $||1|| = \sqrt{2}$

$$< x, x >= \int_{-1}^{1} xx dx = \frac{2}{3}$$

$$||x|| = \sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6}$$

$$< x^2, x^2 >= \int_{-1}^{1} x^2 x^2 dx = \frac{2}{5}$$

$$||x^2|| = \sqrt{\frac{2}{5}} = \frac{1}{5}\sqrt{10}$$

Therefore

An orthonormal basis of P_2 is

$\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \frac{\sqrt{5}}{\sqrt{2}}x^2$

ANOTHER EXAMPLE:

If we take $C[-\frac{\pi}{2}, \frac{\pi}{2}]$ as V and defne

$<f,g> = \int_{-\pi/2}^{\pi/2} fg\,dx$

Take f(x)=1,g(x)=x,h(x)=sinx
May check that these functions are linearly independent.
To find an orthogonal basis of W=span{f,g,h}
v1=1,
v2=x-$\frac{<x,1>}{<1,1>} \cdot 1$

$< x, 1 > = \int_{-\pi/2}^{\pi/2} x \cdot 1dx = \int_{-\pi/2}^{\pi/2} xdx = 0$

v2=x

$$v3 = \sin x - \frac{\int_{-\pi/2}^{\pi/2} x\sin x dx}{\int_{-\pi/2}^{\pi/2} x \cdot x dx} x - \frac{\int_{-\pi/2}^{\pi/2} \sin x dx}{\int_{-\pi/2}^{\pi/2} 1 \cdot 1 dx} \cdot 1 = \sin x - \frac{24}{\pi^3}x$$

$\int_{-\pi/2}^{\pi/2} x\sin x dx = 2$ as on the board

$\int_{-\pi/2}^{\pi/2} x \cdot x dx = \frac{1}{12}\pi^3$ as on the board

$\int_{-\pi/2}^{\pi/2} 1 \cdot 1dx = \pi$

v3= $\sin x - \frac{24}{\pi^3}x$

orthogonal basis

1,x, $\sin x - \frac{24}{\pi^3}x$

For othonormal basis

note

$||1|| = \sqrt{\pi}$

$||x|| = \sqrt{\frac{1}{12}\pi^3}$

$||\sin x|| = \sqrt{\int_{-\pi/2}^{\pi/2} \sin^2 x dx} = \frac{1}{2}\sqrt{2}\sqrt{\pi}$

an orthonormal basis is

$\{\frac{1}{\sqrt{\pi}}, \frac{x}{\sqrt{\frac{1}{12}\pi^3}}, \frac{\sin x}{\frac{1}{2}\sqrt{2}\sqrt{\pi}}\}$

Please work on the assigned exercises in the section 6.7 and post your difficulties in the discussion area.