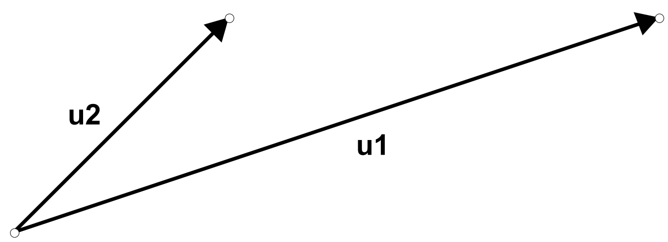
Lesson 28:

In this Lesson, we shall learn the construction of an orthonormal basis from a given basis of a subspace of R^n.

Example 1.

Consider a basis $\{u_1,u_2\}$ of R^2 , where, $u_1=\begin{bmatrix} 3 \\ 1 \end{bmatrix}, u_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Note that this is not an orthogonal basis.



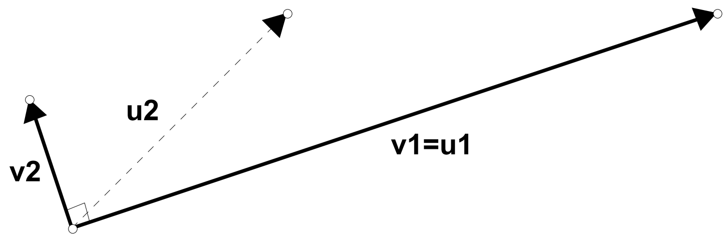
If we replace u_2 by v_2 where v_2 is orthogonal to u_1, then $\{u_1,v_2\}$ is an orthogonal basis of R^2. If we normalize these vectors, we shall get an orthonormal basis.

One way to construct v_2 is to take the orthogonal component $u_2 - \frac{u_2 \cdot u_1}{u_1 \cdot u_1} u_1$

Note that

$$\frac{u_2 \cdot u_1}{u_1 \cdot u_1} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}} = \frac{2}{5}$$

take $v_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{3}{5} \end{bmatrix}$



Therefore, if we took, $v_1=u_1$, then $\{v_1, v_2\}$ is an orthogonal basis. Normalizing v_1 and v_2 will give us an orthonormal basis.

Example 2.

Consider a basis $\{u_1, u_2, u_3\}$ of R^3, where $u_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$,

$u_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Using this basis, let us construct an orthogonal basis of R^3.

Take $v_1=u_1$

and v_2 the component of u_2 orthogonal to $v_1=u_1$, then

$$v_2=u_2 - \frac{u_2 \cdot u_1}{u_1 \cdot u_1} u_1$$

$$\frac{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} = \frac{8}{9}$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \frac{8}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ -\frac{7}{9} \\ \frac{2}{9} \end{bmatrix}$$

then

$$v_1 \cdot v_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{10}{9} \\ -\frac{7}{9} \\ \frac{2}{9} \end{bmatrix} = 0$$ checks that $v_1$and v_2 are orthogonal.

Now since$\{v_1, v_2, u_3\}$ is not an orthogonal set, we construct a vector orthogonal to the subspace Spanned by v_1, v_2.

With u_3 which is not in the span $\{v_1, v_2\}$ available, we may take

$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$\frac{u_3 \cdot v_1}{v_1 \cdot v_1} = \frac{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} = \frac{8}{9}$$

$$\frac{u_3 \cdot v_2}{v_2 \cdot v_2} = \frac{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{10}{9} \\ -\frac{7}{9} \\ \frac{2}{9} \end{bmatrix}}{\begin{bmatrix} \frac{10}{9} \\ -\frac{7}{9} \\ \frac{2}{9} \end{bmatrix} \cdot \begin{bmatrix} \frac{10}{9} \\ -\frac{7}{9} \\ \frac{2}{9} \end{bmatrix}} = \frac{8}{17}$$

$$v_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \frac{8}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{8}{17}\begin{bmatrix} \frac{10}{9} \\ -\frac{7}{9} \\ \frac{2}{9} \end{bmatrix} = \begin{bmatrix} \frac{10}{17} \\ \frac{10}{17} \\ -\frac{15}{17} \end{bmatrix}$$

$$v_2 \cdot v_3 = \begin{bmatrix} \frac{10}{9} \\ -\frac{7}{9} \\ \frac{2}{9} \end{bmatrix} \cdot \begin{bmatrix} \frac{10}{17} \\ \frac{10}{17} \\ -\frac{15}{17} \end{bmatrix} = 0$$

$$v_1 \cdot v_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{10}{17} \\ \frac{10}{17} \\ -\frac{15}{17} \end{bmatrix} = 0$$

Therefore $\{v_1, v_2, v_3\}$ is an orthogonal basis of R^3. Normalizing these basis vectors will give us an orthonormal basis.

Read the Theorem 11 in the section 6.4 to understand the generalization of this process. This process is called the Gram-Schmidt Process.

As another example, look at

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

are basis vectors of R^3.

To find an orthonormal basis
alternate
take

$w_1 = v_1$

$w_2 = v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$

$$w_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix}$$

$w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot v_{w1}} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$

$$w_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix}}{\begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix}} \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

normalize w1,w2,w3

$$w1.w1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$$

$$\widehat{w_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$w2.w2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{8}{3}$$

$$\widehat{w_2} = \frac{3}{\sqrt{8}} \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$w_3 \cdot w_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 2$$

$$\widehat{w_3} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

As another example, let us consider the exercise #8 in section 6.4.

8 and 22

To find an orthonormal basis for span $\left\{ \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} \right\}$

Step 1:

$$\begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} - \frac{\begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}}{\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

An orthogonal basis is

$$\left\{ \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} \right\}$$

to normalize

$$\frac{1}{\sqrt{\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}}} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{10}\sqrt{2} \\ -\frac{2}{5}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$\frac{1}{\sqrt{\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}}} \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}$$

$$\left\{ \begin{bmatrix} \frac{3}{10}\sqrt{2} \\ -\frac{2}{5}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{bmatrix} \right\}$$

an orthonormal basis.

Please work on the Exercises in the section 6.4 and post your questions in the Discussion area.