Lesson 27:

In this lesson, we shall briefly discuss the usage of the orthogonal decomposition theorem in section 6.3.

As an example, let us look at the exercise # 2 in section 6.3 in both the editions.

2. We have the vectors $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$,

$u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $u_4 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$

It is given that the subset containing the above four vectors is an orthogonal basis of R^4 (we can easily verify this.)

We have to write $v=\begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$ as a sum of two vectors, one in Span$\{u_1\}$ and the other in Span$\{u_2, u_3, u_4\}$

the prjection in Span$\{u_1\}$ is $\frac{v \cdot u_1}{u_1 \cdot u_1} u_1 = 2\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}$

the prjection in the Span$\{u_2, u_3, u_4\}$ is $= \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$

Therefore the desired decomposition is $\begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$

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an example, let us look at the exercise # 10 in section 6.3 in both the editions.

Given vectors $y=\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$, $u_1=\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $u_2=\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $u_3=\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. Let $W=\{u_1,u_2,u_3\}$. Note that W is an orthogonal subset.

We have to write y as a sum of two vectors, one in W and the other in $W^\perp$.

Write $y=\hat{y}+z$, where $\hat{y}$ is in W and z is in $W^\perp$.

By the Orthogonal Decomposition Theorem,

$$\hat{y}=\frac{y \cdot u_1}{u_1 \cdot u_1}u_1+\frac{y \cdot u_2}{u_2 \cdot u_2}u_2+\frac{y \cdot u_3}{u_3 \cdot u_3}u_3 \qquad \text{and } z=y-\hat{y}$$

Note that

$$\frac{y \cdot u_1}{u_1 \cdot u_1}=\frac{\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}} = \frac{1}{3}$$

$$\frac{y \cdot u_2}{u_2 \cdot u_2} = \frac{\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}} = \frac{14}{3}$$

$$\frac{y \cdot u_3}{u_3 \cdot u_3} = \frac{\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}} = -\frac{5}{3}$$

Therefore

$$\hat{y} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{14}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

$$z = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$