Lesson 26

Before reading this lesson, please read the in the section 6.2: the definition of an orthogonal subset of R^n

The example 1

Definition of an orthogonal basis.

Theorem 5

The following example will demonstrate the idea in the above part.
Note: We have done some of the following calculations by using much easier methods before. The following calculations will help us understand the further discussions in a slighly abstract setting more easily.

Let us take up the exercise #10 in the section 6.2 in both the second updated and the third edition.

We are given the vectors $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

we have to
a) show that B= $\{u_1, u_2, u_3\}$ is an orthogonal basis of R^3.
b) express x=$\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ as a linear combination of the elements of B=$\{u_1, u_2, u_3\}$.

a) Note that $u_1 \cdot u_2 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 3 \times 2 + (-3) \times 2 + 0 \times (-1) = 0$

$u_2 \cdot u_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 2 \times 1 + 2 \times 1 + (-1) \times 4 = 0$

$u_3 \cdot u_1 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = 1 \times 3 + 1 \times (-3) + 4 \times 0 = 0$

Therefore B is an orthogonal subset of R^3 and is lenearly idependent. Since B has 3 linearly vectors in it, B is a basis of R^3, which obviously is orthogonal.

b) We have to express x as a linear combination of the elements of $B=\{u_1, u_2, u_3\}$. i.e. have to find c_1, c_2, and c_3 such that $x=c_1u_1+c_2u_2+c_3u_3$. Note that according to the theorem 5 of this section,

$c_1=\frac{x \cdot u_1}{u_1 \cdot u_1}$ $c_2=\frac{x \cdot u_2}{u_2 \cdot u_2}$ $c_3=\frac{x \cdot u_3}{u_3 \cdot u_3}$

$$c_1=\frac{\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}} = \frac{24}{18} = \frac{4}{3}$$

$$c_2=\frac{\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}} = \frac{1}{3}$$

$$c_3 = \frac{\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}} = \frac{1}{3}$$

therefore

$$\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = \frac{4}{3}\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Orthogonal Projections:

Read the discussion on the page 386 in the third edition and page 381 in the sceond updated edition.

the summary is that if u is a nonzero vector in R^n .
For a vector $y \in R^n$

$\hat{y} = \frac{y \cdot u}{u \cdot u} u$ is the projection of y in the direction of u
and $z = y - \frac{y \cdot u}{u \cdot u} u$ is the projection of y in the direction orthogonal to u.

Example:

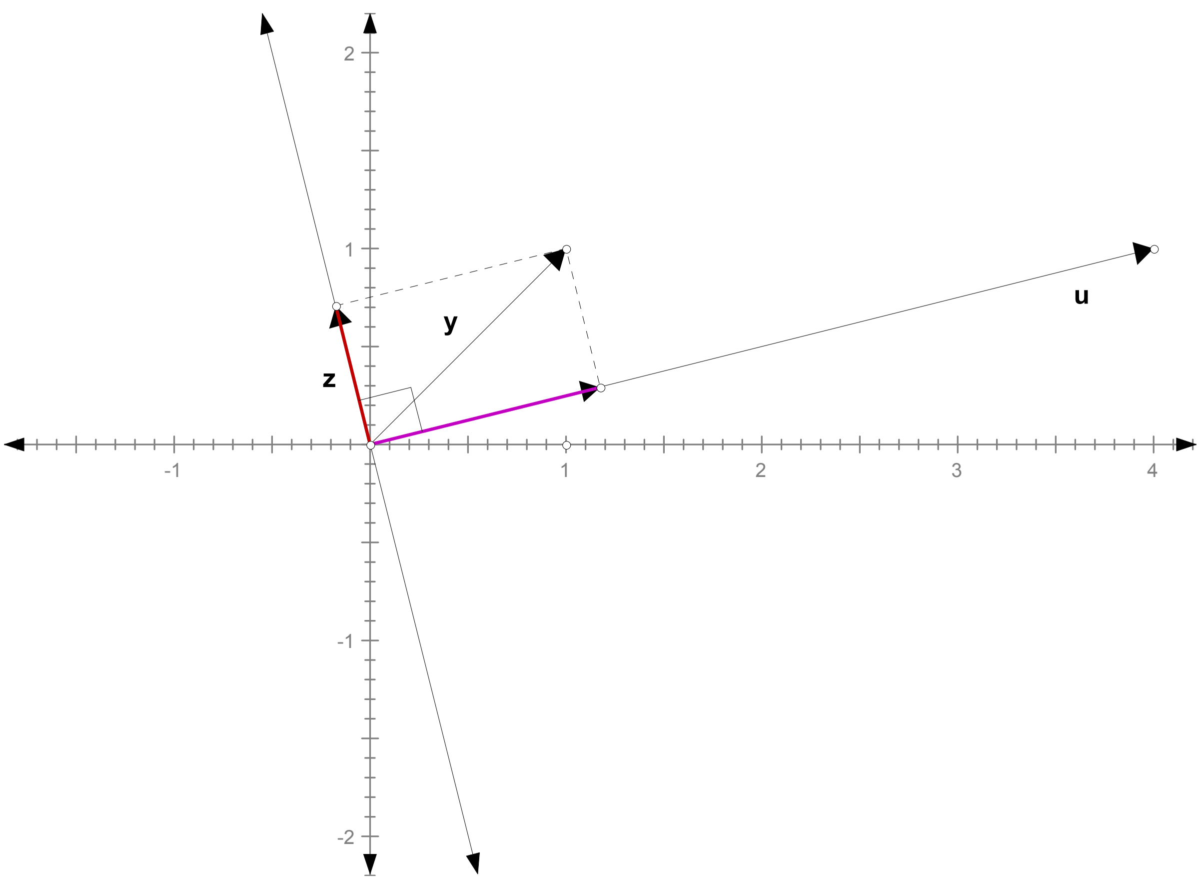
Consider $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in R^2

$$\hat{y} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}}{\begin{bmatrix} 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{20}{17} \\ \frac{5}{17} \end{bmatrix}$$

$$z = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{20}{17} \\ \frac{5}{17} \end{bmatrix} = \begin{bmatrix} -\frac{3}{17} \\ \frac{12}{17} \end{bmatrix}$$

geometrically

we have the following picture with $\hat{y}$ as the purple vector.



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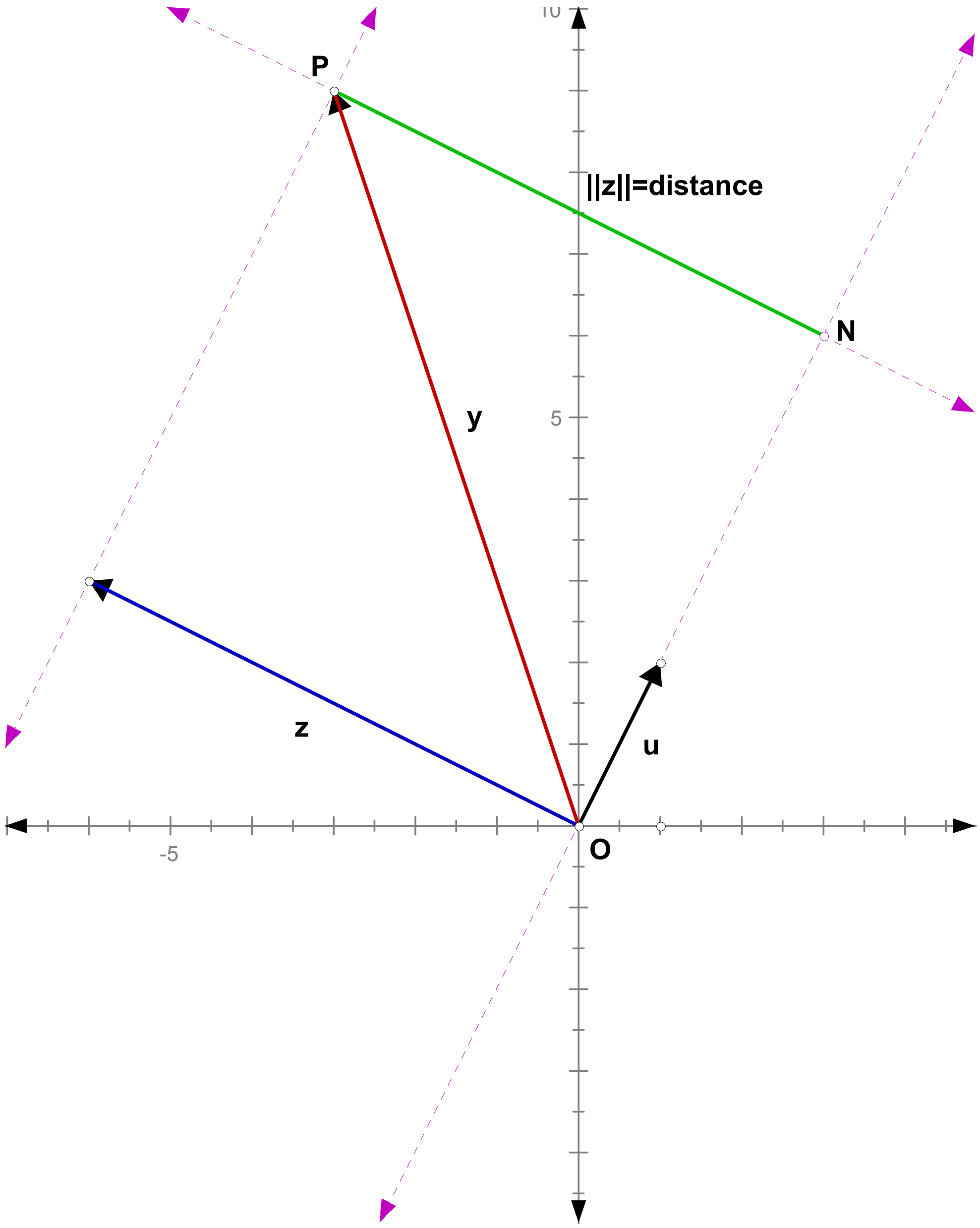
As another example, let us look at exercise #16 in section 6.2 in both the third edition and the second updated edition.

We have to compute the distance from y$=\begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and the line through u$=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the origin.

As the following picture shows, the ditance is the $\|z\|$, where z=y$-\frac{y \cdot u}{u \cdot u}$u

$$z=\begin{bmatrix} -3 \\ 9 \end{bmatrix} - \frac{\begin{bmatrix} -3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

$\|z\| = \left\| \begin{bmatrix} -6 \\ 3 \end{bmatrix} \right\| = \sqrt{(-6)^2 + 3^2} = 3\sqrt{5}$ units



Orthonormal Sets:

An orthogonal set of vectors S in R^n is orthonormal if each of the vectors in S is a unit vector.

Recall that normailizing a vector u means creating a unit vector in the direction of u.

As an illustration, let us look at exercise #20 in the section 6.2 in the third edition as well as the second updated edition.

We are given the vectors $u_1 = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, u_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix}$.

We have to check if the set formed by the above two vectors is orthonormal.
If it is orthogonal but is not orthonormal then we have to normalize the vectors to produce an orthonormal set.

Since,

$$\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} = 0$$

the vectors are orthogonal

$$\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = 1$$

$$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} = \frac{5}{9}$$

therefore $||u_2|| = \sqrt{\frac{5}{9}} = \frac{1}{3}\sqrt{5}$

$u_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix}$ is not a unit vector.

We can normalize u_2 by taking

$$\frac{1}{\|u_2\|}u_2$$

$$=\frac{1}{\frac{1}{3}\sqrt{5}}\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} \\ 0 \end{bmatrix}$$

the new set, formed by the vectors

$$\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} \\ 0 \end{bmatrix}$$ is orthonormal.

Just as another illustration, As an illustration, let us look at exercise #22 in the section 6.2 in the third edition as well as the second updated edition.

We have

$$\begin{bmatrix} \frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix} \bullet \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \bullet \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} = 0$$

$$\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \bullet \begin{bmatrix} \frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix} \bullet \begin{bmatrix} \frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix} = 1$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \bullet \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = 1$$

$$\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \bullet \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} = 1$$

the above equations show that the three vectors are orthogonal and each of them is a unit vector. This shows that the set formed by these vectors is an orthonormal set.

Please finish the exercises in the section 6.2 and post the difficulties in the discussion area.