In this lesson, we shall start chapter 6 in which we shall study Inner Products.

Fisrt section takes up the dot product in R^n which is an example of an inner product, later we shall take up definition of
of inner products for vector spaces in general.

Recall the definition of the dot product.

Example 1:

If $u=\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$, $v=\begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$

The dot product u·v=

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix} = 1 \times 2 + 2 \times 1 + (-1) \times 3 + 3 \times (-1) = -2$$

Note that the value of the dot product of two vectors is a scalar.

The norm of u is

$$||u|| = \sqrt{u \cdot u} = \sqrt{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}} = \sqrt{1 \times 1 + 2 \times 2 + (-1) \times (-1) + 3 \times 3} = \sqrt{15}$$

Example 2:

Recall from the definition that a unit vector is a vector with norm 1.

To find a unit vector in the direction of a non zero vector u, we compute $\frac{1}{||u||}$u.

Take u= $\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$,

||u||= $\sqrt{2^2 + 4^2 + 4^2} = 6$

Therefore a unit vector in the direction of u is given by

$$\frac{1}{6}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

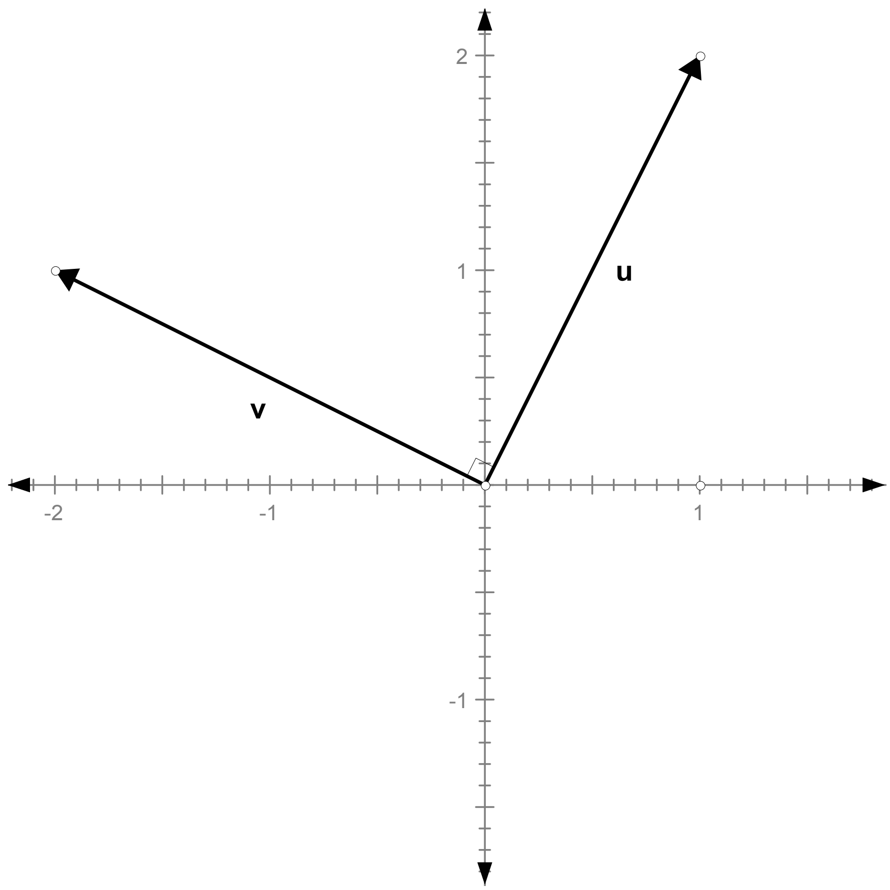
Example 3:

Recall the definition that the vectors u and v are orthogonal if u·v=0

Take u= $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and v= $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

u·v= $1 \times (-2) + 2 \times 1 = 0$

Geometrically:



If W is a subspace of R^n, recall the definition of $W^{\perp}$=={x∈ R^n such that x•u=0 for all u in W}

Example 4.

Take W= $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \text{ are real numbers} \right\}$. W is a subspace of R^3.

Note that $W^{\perp}$= $\left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : \text{z is a real number} \right\}$

Example 5:

Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a subspace of R^4.

Show that $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in W^{\perp}$

Note that

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = 0$$

therefore $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is orthogonal to both the basis elements of W, therefore is orthogonal to all the elements of W.

Discussion

Let us look at #30 in section 6.1 of both the second edition and the third edition.

We are given that W is a subspace of R^n and we have to prove that $W^{\perp} = \{x \in R^n$ such that $x \cdot u = 0$ for all u in W$\}$ is a subspace of R^n.

Let us follow the steps suggested in the exercise.

a. $z \in W^{\perp} \Rightarrow w \cdot z=0$ for each $w \in W$,
therefore for any scalar c, $(cz) \cdot w=c(z \cdot w)=c(0)=0$, for all $w \in W$.

b. $z_1, z_2 \in W^{\perp}$
$\Rightarrow w \cdot z_1 = 0$ and $w \cdot z_2 = 0$ for each w in W.
$\Rightarrow w \cdot (z_1+z_2)= w \cdot z_1+w \cdot z_2 = 0 + 0 = 0$ dot product is distributive over +
$\Rightarrow z_1+z_2 \in W^{\perp}$

c. The only requirement that remains to be verified is that $0 \in W^{\perp}$ (generally we check it first.)
This is obviously true because $0 \cdot w=0$ for all $w \in W$.

Therefore $W^{\perp}$ is a subspace of R^n.

Discussion

Let us look at #31 in section 6.1 of both the second edition and the third edition.

We have to show that

$w \in W \cap W^{\perp} \quad \Rightarrow \quad w=0$

Proof

$w \in W \cap W^{\perp}$
$\Rightarrow w \in W$ and $w \in W^{\perp}$
$\Rightarrow w \cdot w=0$
$\Rightarrow w=0$

Please work on the exercises in the section 6.1 and post your questions in the discussion area.