Lesson 24

Recall that to find the Eigen values of a square matrix A, we look at the solutions of the equation

$|A - \lambda I| = 0$.

Let us work on the exercise number 6 in section 5.2 in the second updated edition as well as in the third edition.

To find the Eigen values of $\begin{bmatrix} 3 & -4 \\ 4 & 8 \end{bmatrix} = A$

look at

$$\left| \begin{bmatrix} 3 & -4 \\ 4 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

or $\left| \begin{bmatrix} 3 & -4 \\ 4 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$

or $\begin{vmatrix} 3-\lambda & -4 \\ 4 & 8-\lambda \end{vmatrix} = 0$

or $(3-\lambda)(8-\lambda) - (-4)(4) = 0$

0r $\lambda^2 - 11\lambda + 24 + 16 = 0$

0r $\lambda^2 - 11\lambda + 40 = 0$

$\frac{11 \pm \sqrt{(-11)^2 - 160}}{2} = \frac{11}{2} \pm \frac{1}{2} i\sqrt{39}$

$\begin{bmatrix} 3 & -4 \\ 4 & 8 \end{bmatrix}$, eigenvalues: $\frac{11}{2} + \frac{1}{2} i\sqrt{39}, \frac{11}{2} - \frac{1}{2} i\sqrt{39}$

Let us work on the exercise number 18 in section 5.2 in the second updated edition as well as in the third edition.

$$A = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$$

Characterstic polynomial

$$\begin{vmatrix} -7-\lambda & -16 & 4 \\ 6 & 13-\lambda & -2 \\ 12 & 16 & 1-\lambda \end{vmatrix} = -75 + 5\lambda + 7\lambda^2 - \lambda^3$$

synthetic Division by $(\lambda - 5)$

$$\begin{array}{rrrrr} & -1 & 7 & 5 & -75 \\ 5 & \downarrow & -5 & 10 & 75 \\ & -1 & 2 & 15 & 0 \end{array}$$

$-75 + 5\lambda + 7\lambda^2 - \lambda^3 = (\lambda - 5)(-\lambda^2 + 2\lambda + 15) = (\lambda - 5)(5 - \lambda)(\lambda + 3)$

Therefore the Eigen Values of

$\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$ are 5,5,-3

To find an Eigen Vector X corresponding to 5,
Look at the solutions of

$(A - 5I)X = 0$

$$\begin{bmatrix} -12 & -16 & 4 \\ 6 & 8 & -2 \\ 12 & 16 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} -12 & -16 & 4 & 0 \\ 6 & 8 & -2 & 0 \\ 12 & 16 & -4 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & \frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of the Eigen space of 5.

To find the Eigen Vectors of -3, row reduce

$$\begin{bmatrix} -4 & -16 & 4 & 0 \\ 6 & 16 & -2 & 0 \\ 12 & 16 & 4 & 0 \end{bmatrix}$$, row echelon form: $$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$ is a basis of THe Eigen Space of -3.

Now form the matrix $\begin{bmatrix} -1 & -\frac{4}{3} & \frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ by using the above basis vectors and note that

$$\begin{bmatrix} -1 & -\frac{4}{3} & \frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix} \begin{bmatrix} -1 & -\frac{4}{3} & \frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$ is similar to the diagonal matrix $\begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

Please look at the example 3 (solved) of section 5.3 in both editions for step by step approach of

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Let us work on the exercise number 12 in section 5.2 in the second updated edition as well as in the third edition.

To attempt to diagonalize $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ with eigenvalues: $8,2,2$

given to us.

For 8

$\begin{bmatrix} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis for the Eigenspace of the eigen value 8 is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

For 2

$\begin{bmatrix} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a basis for ES of 2 is, note

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Note that we have done

1. How to find the charactorstic polynomial of a square matrix.
2. How to use the charactorstic polynomial to finf the Eigen

Values of a square matrix.
3. How to find the eigen vectors for given Eigen Values.
4. Have seen that the subset E_λ that contains the zero vector and the Eigen Values of λ is a subspace.
5. How to diagonalize a square matrix A (if possible) using the eigen values.

To review the procedure

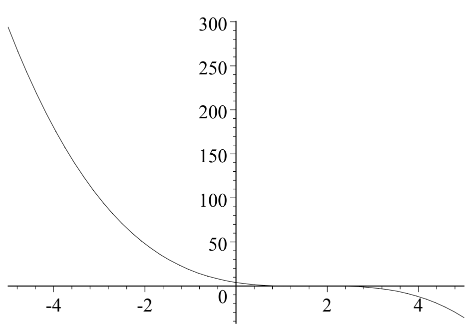
Let us work on the exercise number 12 in section 5.2 in the second updated edition as well as in the third edition.

Doing More than the problem is asking, even though we are given the eigen values, still we shall see how to obtain them if they were not given.

$$A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$$

To find eigen values

Look at $|A - \lambda I| = \begin{vmatrix} -\lambda & -4 & -6 \\ -1 & -\lambda & -3 \\ 1 & 2 & 5-\lambda \end{vmatrix} = 5\lambda^2 - \lambda^3 - 8\lambda + 4$



$-\lambda^3 + 5\lambda^2 - 8\lambda + 4$

Looks like $(\lambda - 2)$ is factor of $-\lambda^3 + 5\lambda^2 - 8\lambda + 4$

Used the synthetic division to see

$\frac{-\lambda^3+5\lambda^2-8\lambda+4}{\lambda-2} = -\lambda^2+3\lambda-2$

$-\lambda^3+5\lambda^2-8\lambda+4 = -(\lambda-2)(\lambda^2-3\lambda+2) = -(\lambda-2)(\lambda-2)(\lambda-1)$

Eigen values are 2,2,1

or 2,1

To find the eigen vectors corresponding to 2

Look at

$\begin{bmatrix} -2 & -4 & -6 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a basis for the Eigen space of 2 is

$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

For eigen vectors of 1, look at

$\begin{bmatrix} -1 & -4 & -6 & 0 \\ -1 & -1 & -3 & 0 \\ 1 & 2 & 4 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a basis is

$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

Look at P= $\begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To review the procedure

Let us work on the exercise number 26 in section 5.1 in the second updated edition as well as in the third edition.

26.

λ is an eigen value of A

$\rightarrow AX=\lambda X$ for some nonzero X

$\rightarrow AAX=\lambda AX$

$\rightarrow 0=\lambda AX$ because $A^2=0$

$\rightarrow 0=\lambda\lambda X$

$\rightarrow 0=\lambda^2 X$

$\rightarrow \lambda^2 = 0$ because X is nonzero

$\rightarrow \lambda = 0$

Therefore 0 is the only eigen value.

Let us work on the exercise number 27 in section 5.1 in the second updated edition as well as in the third edition.

27. Keep in mind that we have a square matrix.

Recall that A^T is the transpose of A, obtained by changing rows to columns and vice versa.

Note that

$(A-\lambda I)^T = A^T - \lambda I$

$|(A-\lambda I)|$

$=|(A-\lambda I)^T|$ because Changing rows to columns does not affect the value of the determinant.

$=|A^T - \lambda I|$

Therefore A and A^T have the same characetorstic polynomial.

Therefore A and A^T have the same eigen values.

QED

or

λ is an eigen value of A

$\Leftrightarrow$ λ is a root of the equation $|(A - \lambda I)| = 0$

$\Leftrightarrow$ λ is a root of the equation $|(A^T - \lambda I)| = 0$

$\Leftrightarrow$ λ is an eigen value of A^T.

QED

Finish 5.1,5.2,5.3 please

and post your difficulties in the discussion area.