Lesson 23:

Now we are starting on chapter 5.

Suppose A is a square matrix and λ is a scalar. λ is called an eigen value of A if there is a non zero vector v such that $Av = \lambda v$.
v is called an eigenvector of A corresponding to λ.

Example 1:

Let $A = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$ and $\lambda = 1$

a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigen vector corresponding to 1 if

$$\begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\rightarrow$

$$\begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\rightarrow$

$$\begin{bmatrix} .6x_1 + .3x_2 \\ .4x_1 + .7x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\rightarrow$

$$\begin{bmatrix} .6x_1 + .3x_2 - x_1 \\ .4x_1 + .7x_2 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightarrow$

$$\begin{bmatrix} -.4x_1 + .3x_2 \\ .4x_1 - .3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives

$$\left.\begin{array}{r} -.4x_1 + .3x_2 = 0 \\ .4x_1 - .3x_2 = 0 \end{array}\right\} \quad \Rightarrow \quad x_1 = \frac{3}{4}x_2$$

treating x_2 as a free variable, note that $\begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$

check

$$\begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

therefore $\begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 1.

Note that $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is another eigen vector of A corresponding to the eigenvalue 1.

$$\begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3.0 \\ 4.0 \end{bmatrix}$$

Example 2.

Given $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and $\lambda = 2.$

To find the Eigen Vectors corresponding to 2.

Are looking for $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 + 2x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + 2x_3 \end{bmatrix} - 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_2 + 2x_3 \\ x_1 - x_2 + x_3 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

The only possibility is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, which is not an acceptable Eigen Vector because it is not different from the zero vector.

Therefore no Eigen Value for 2 with respect to A.

Let us work on the exercise number 6 in section 5.1 in the second updated edition as well as in the third edition.

$A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$ to check if $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an Eigen Vector of A.

Remember, $X \neq 0$ is an eigen vector of A
iff $AX = \lambda X$ for some real number λ.

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Yes, the given vector is an Eigen vector and the corresponding Eigen Value is -2.

Note that v is an eigenvector of a matrix A if Ax is a multiple of v.

Let us develop a method to obtain the eigen values for a square matrix A.

λ is an Eigen Value of $A_{n\times n}$
$\rightarrow$ we have $v\neq 0$ such that $Av=\lambda v$ .. want to solve for λ
$\rightarrow Av-\lambda Iv=0$ where I is the $n\times n$ identity matrix.
$\rightarrow (A-\lambda I)v=0$ want λ, know that a non trivial v exists.
$\rightarrow (A-\lambda I)$ is not invertible
$\rightarrow |A-\lambda I|=0$

May also go back step by step.
Therefore to obtain the eigen values λ of a square matrix A, solve $|A-\lambda I|=0$ for λ.

therefore λ is an Eigen Value of A $\Leftrightarrow$ λ is a solution of the equstion $|A-\lambda I|=0$

The polynomial $|A-\lambda I|$ is called the chracterstic polynomial of A.

The equation $|A-\lambda I|=0$ is called the characeterstic equation of A.

For an eigenvalue λ, note that the set of the vecors v such that $(A-\lambda I)v=0$ is the null space of A. If A is $n\times n$, then the solutions of $(A-\lambda I)v=0$ is a subspace of R^n and is called the eigen space of A corresponding to the eigenvalue λ.

or briefly

For an eigen value λ of a square matrix A, the solutions of $(A - \lambda I)v=0$ form a subspace, called e

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Let us work on the exercise number 6 in section 5.1 in the second updated edition as well as in the third edition.

Given $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$

and given that an Eigen value is -2. We have to find a basis for the Eigen Space of -2.

Eigen Space of -2 is the set of solutions of

$(A+2I)X = 0$

i.e. we are looking for the solutions of the above system of homogeneous equations.

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix} = A + 2I$$

The elements of the Eigen Space of -2 are the solutions corresponding to the following aug matrix

$$\begin{bmatrix} 3 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & -13 & 3 & 0 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

A basis is $\left\{ \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right\}$

check the answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -2 \end{bmatrix} = -2\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$ we did good.

Let us work on the exercise number 16 in section 5.1 in the second updated edition as well as in the third edition.

To find the Eigen space of 4 for the matrix $\begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = A$

Look at the solution space of
$(A - 4I)X = 0$
or

$$\begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Shall look at

$\begin{bmatrix} -1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

A basis $\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Check

$$\begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

Read the theorem and the proof of the theorem 2 in section 5.1, which states that the eigen vectors corresponding to distinct eigen values are independent.

Please work on the exercises in the section 5.1
and post your difficulties in the discussion area.