Lesson 22:

Covers section 4.9:

An application:

Suppose that we have two long distance companies in a region.
A and O
month to month, the switching rate between the two companies is the following
40% of A's subscribers stay with A
60% of A's subscribers switch to O
90% of O's subscribers stay with O
10% of O's subscribers switch to A.

At this time A has 80% of the market and B has 20% of the market.
Calculate the next month's share of A and O using basic arithmetic
A: $.4 \times .8 + .1 \times .2 = .34$
B: $.6 \times .8 + .9 \times .2 = .66$
But note that we can take

$$T = \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}$$ **As the the transition matrix from one month to another**

$$\hookleftarrow \quad A \quad O$$

$$\mathbf{T} = \begin{matrix} A \\ O \end{matrix} \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}$$

This Month, the month 1.

$$X_1 = \begin{bmatrix} .8 \\ .2 \end{bmatrix} \begin{matrix} A \\ O \end{matrix}$$

In the month 2,

$$\mathbf{T}X_1 = \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix} \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .34 \\ .66 \end{bmatrix} \begin{matrix} A \\ O \end{matrix}$$

Suppose this trend continues

$$X_2 = \begin{bmatrix} .34 \\ .66 \end{bmatrix} \begin{matrix} A \\ O \end{matrix}$$

$$X_3 = TX_2 = \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix} \begin{bmatrix} .34 \\ .66 \end{bmatrix} = \begin{bmatrix} .202 \\ .798 \end{bmatrix} \begin{matrix} A \\ O \end{matrix}$$

$$X_4 = TX_3 = \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}\begin{bmatrix} .202 \\ .798 \end{bmatrix} = \begin{bmatrix} .1606 \\ .8394 \end{bmatrix} \begin{matrix} A \\ O \end{matrix}$$

$$X_5 = TX_4 = \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}\begin{bmatrix} .1606 \\ .8394 \end{bmatrix} = \begin{bmatrix} .14818 \\ .85182 \end{bmatrix} \begin{matrix} A \\ O \end{matrix}$$

$$X_6 = TX_5 = \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}\begin{bmatrix} .14818 \\ .85182 \end{bmatrix} = \begin{bmatrix} .144454 \\ .855546 \end{bmatrix} \begin{matrix} A \\ O \end{matrix}$$

....

Also note that

$$X_6 = T^5 X_1 = \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}^5 \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .144454 \\ .855546 \end{bmatrix} = X_6$$

$$X_{12} = \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}^{11} \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .142858 \\ .857142 \end{bmatrix}$$

Warning:

Some times the starting matrix is denoted by X_o rather than X_1.

Question: Will this ever stablize

i.e.

$$\begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}\begin{bmatrix} .142858 \\ .857142 \end{bmatrix} = \begin{bmatrix} .142857 \\ .857143 \end{bmatrix}$$

i.e., can we get a vector $\begin{bmatrix} p \\ q \end{bmatrix}$ such that

$$\begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

i.e.

$$\begin{bmatrix} .4p + .1q \\ .6p + .9q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

i.e.

$.4p + .1q = p \rightarrow -.6p + .1q = 0$

$.6p + .9q = q \rightarrow .6p - .1q = 0$

In this context, we have $p + q = 1$

$p + q = 1$

$-.6p + .1q = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ -.6 & .1 & 0 \end{bmatrix}$$, row echelon form: $$\begin{bmatrix} 1 & 0 & .142857 \\ 0 & 1 & .857143 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -\frac{3}{5} & \frac{1}{10} & 0 \end{bmatrix}$$, row echelon form: $$\begin{bmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 1 & \frac{6}{7} \end{bmatrix}$$

This means that eventually, A will have $\frac{1}{7}^{th}$ and O will have $\frac{6}{7}^{th}$ of the market.

Let us just look at the sequence

$$A_k = \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}^k$$

$$\begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}^{100} = \begin{bmatrix} .142857 & .142857 \\ .857143 & .857143 \end{bmatrix}$$

$$\begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix}^{200} = \begin{bmatrix} .142857 & .142857 \\ .857143 & .857143 \end{bmatrix}$$

Stochastic Matrix T,
shows the transition from one stage to another,
has the follwoing properties
1. All the entries are non negative
2. The sum of the entries in each column is 1.

State Matrix I
Is a h column matrix such that
1. all the entries are non negative
2. sum of the entries is 1.

In a given context if T is an nxn matrix, then I will be a nx1 matrix.

If I_0 is the initial stage
then the kth stage is given by $I_k=T^kI_0$

Also, we will get a steady state I_S such that
$TI_S=I_S$

Note from our above computations, that $I_S$does not depend on

I_0.

More examples from the text:

3. page 296 in the third edition or 2. page 290 in the second updated edition.

$$\begin{array}{c} \hookleftarrow \quad H \quad I \\ \begin{matrix} H \\ I \end{matrix} \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \end{array}$$ Is the stchotastic Matrix of the context

b)

Today the Monday

$$\begin{bmatrix} .80 \\ .20 \end{bmatrix} \begin{matrix} H \\ I \end{matrix}$$

$$\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} .80 \\ .20 \end{bmatrix} = \begin{bmatrix} .85 \\ .15 \end{bmatrix}$$ Tuesday

$$\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} .85 \\ .15 \end{bmatrix} = \begin{bmatrix} .875 \\ .125 \end{bmatrix}$$ Wednesday

$$\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}^2 \begin{bmatrix} .80 \\ .20 \end{bmatrix} = \begin{bmatrix} .875 \\ .125 \end{bmatrix}$$

$.95 \times .05 + .05 \times .55 = .075$ sick two days from now

$1 - .0725 = .9275$ well two days from now

$$\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} = \begin{bmatrix} .95 \times .95 + .45 \times .05 & .95 \times .45 + .45 \times .55 \\ .05 \times .95 + .55 \times .05 & .05 \times .45 + .55 \times .55 \end{bmatrix} =$$

$$\begin{bmatrix} .925 & .675 \\ .075 & .325 \end{bmatrix}$$

Extend to our own,

If the same pattern continues, what is probability that a person healthy today will be sick in four days from today

$$\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}^4 = \begin{bmatrix} .90625 & .84375 \\ .09375 & .15625 \end{bmatrix}$$

answer: $.09375$

Stochastic, each entry is non negative, entries in each column add to 1.
Regular Stochastic: If some power of P contains strictly positive entries.

$$\begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}^2 = \begin{bmatrix} 1 & .36 \\ 0 & .64 \end{bmatrix}$$

$$\begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}^3 = \begin{bmatrix} 1 & .488 \\ 0 & .512 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} * & * \\ 0 \cdot a + d \cdot 0 & * \end{bmatrix}$$

$\begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}$ is not regular

---

Check which of the following are regular and which are not.

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} * & * \\ * & cb + 0 \cdot 0 \end{bmatrix}$$

$$\begin{bmatrix} .6 & 1 \\ .4 & 0 \end{bmatrix}^2 = \begin{bmatrix} .76 & .6 \\ .24 & .4 \end{bmatrix}$$

$$\begin{bmatrix} .6 & 0 \\ .4 & 1 \end{bmatrix}^2 = \begin{bmatrix} .36 & 0 \\ .64 & 1.0 \end{bmatrix}$$

$$\begin{bmatrix} .6 & 0 \\ .4 & 1 \end{bmatrix}\begin{bmatrix} .6 & 0 \\ .4 & 1 \end{bmatrix} = \begin{bmatrix} * & .6 \cdot 0 + 0 \cdot 1 \\ * & * \end{bmatrix}$$

$$\begin{bmatrix} 0 & .6 \\ 1 & .4 \end{bmatrix}^2 = \begin{bmatrix} .6 & .24 \\ .4 & .76 \end{bmatrix}$$

Please work on the section 4.9 exercises and post your difficulties in the discussion area.