Lesson 20:

Recall

Definition:
If a basis of a vector space V has n elements then we call n the dimension of V.

Theorem: If one basis of V has n elements then each basis of V will have n elements.

Definition: If the number of elements in a basis of a vector space V is infinite, then we call V infinite dimensional.

The Basis Theorem:
A linearly independent subset of a vector space V is either a basis of V or it can be extended to be a basis of V.

Examples from the text:

Let us first look at
6 on the page 261 in the third edition, which gives us the subspace $\left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -3a+b+c \end{bmatrix} : a,b,c \text{ in } R^3 \right\}$ of R^3 and we have to find
a) Basis
b) State the dimension.

Note that is purely a review.
a)

$$\begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -3a+b+c \end{bmatrix} = a\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} + b\begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Therefore the above subspace is spanned by

$$\left\{ \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Since

$\begin{bmatrix} 3 & 6 & -1 \\ 6 & -2 & -2 \\ -3 & 1 & 1 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

shows that the third column is a linear combination of the first two, which are pivot columns, there for

by the spanning theorem, $\left\{ \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis of the

above subspace.

b)

the dimension of the subspace is 2 because the number of elements in a basis is 2.

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22 on page 261 in the third edition.

Given the polynomials

B=$\{1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3\}$

To show that B is a basis of P_3.

Have to show

i) B is linearly independent

ii) B spans P_3

Details from scratch

i)

For linear independence must show

$a \cdot 1 + b(1 - t) + c(2 - 4t + t^2) + d(6 - 18t + 9t^2 - t^3) = 0$

$\rightarrow a = b = c = d = 0$

Let us see

$a \cdot 1 + b(1 - t) + c(2 - 4t + t^2) + d(6 - 18t + 9t^2 - t^3) = 0$

collect the like terms and note that the above implies

$(a + b + 2c + 6d) + (-b - 4c - 18d)t + (c + 9d)t^2 - dt^3 = 0$

$$\rightarrow \begin{cases} a + b + 2c + 6d = 0 \\ -b - 4c - 18d = 0 \\ c + 9d = 0 \\ d = 0 \end{cases}$$

Now note by going from bottom to top that

$d = 0 \rightarrow c = 0 \rightarrow b = 0 \rightarrow a = 0$ IN THIS PARTICULAR CASE

OR may also note that from the coordinate vectors that

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

each column is a pivot column, therefore the polynomials are linearly independent.

ii)

Now to see that

$P_3 = span(B)$

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ecah row has a pivot position therefore the coordinate vectors of the polynomials in B span R^4 and therefore the polynomials in B span P_3.

DONE:

Here is the rationale of why the procedure in part b) of the above is valid.

take any polynomial $(a_o + a_1 t + a_2 t^2 + a_3 t^3)$ in P_3,

we should be able to find x,y,z,w such that

$x \cdot 1 + y(1-t) + z(2-4t+t^2) + w(6-18t+9t^2-t^3) = a_o + a_1 t + a_2 t^2 + a_3 t^3$ for all a_o, a_1, a_2, a_3 in R

or

$(x+y+2z+6w)+(-y-4z-18w)t+(z+9w)t^2-wt^3 = a_o+a_1 t+a_2 t^2+a_3$

therefore

$x + y + 2z + 6w = a_o$

$-y - 4z - 18w = a_1$

$z + 9w = a_2$

$-w = a_3$

or

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a_o \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$ has to have solutions for all $\begin{bmatrix} a_o \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \in R^4$

Each row of the coefficient matrix has to have a pivot position.

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$, row echelon form: $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ecah row has a pivot position therefore answer is YES.

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24. on page 261 in the third edition:

We have to find the coordinate vector of p$(t)=7-8t+3t^2$ relative to the basis B=$\{1, 1-t, 2-4t+t^2\}$.
Note that

$7-8t+3t^2$ → coordinate vector $\begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix}$

1 → coordinate vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$1-t$ → coordinate vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$2-4t+t^2$ → coordinate vector $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$

Looking in terms of the coordinate vectors, we have to find the coordinates of $\begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix}$

relative to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$.

For which we look at the solution of the linear system corresponding to the augmented matrix

$\begin{bmatrix} 1 & 1 & 2 & 7 \\ 0 & -1 & -4 & -8 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

therefore $[\mathbf{7-8t+3t^2}]_B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

note that

$5(1)-4(1-t)+3(2-4t+t^2) = 7-8t+3t^2$

Please work on the sction 4.5 exercises
Section 4.6 is self study.

Post your difficulties in the discussion area.