Lesson 2:
This lesson is mainly devoted to vocabulary which will be very important to understand the upcoming ideas.

First recall from the lesson 1 that
The row operations on a matrix are

1. Interchanging any two rows
2. Multiplying a row by a non zero number
3. Adding a row with a multiple of another row
Remember that we need the text book by our side to understand the lesson.
Part I of this session:

In this part, we shall understand the terms, leading entry, echelon form, and row reduced echelon form.

IMPORTANT: Before reading the examples below, please read the definitions of a leading entry, echelon form, and reduced echelon form on the page 14 of the text book (same page in both the editions) .

I am reproducing the definitions from the text, because the book store is still waiting for additional copies.)

A leading entry is the leftmost non zero entry in a row.

Definition:

A matrix is in echelon form if
1. All the rows that have all zeros are below the rows with any non zero numbers in it.
2. The leading entry in a row is to the right the leading entry in the row above it.
3. All the entries in a column below a leading entry are zero.

In order to be in a reduced echelon form, a matrix must have the following two additional conditions.

4. A leading entry must equal 1.
5. All the entries other than the leading in a column containg a leading entry must be zeros.

Note that the conditions 1 thru 3 in the relate to an echelon form.

And we need additional conditions 4 and 5 on an Echlon Form for it to be a reduced Echelon Form.

I hope that the following example will make it clear.
Examples:

1. Is the matrix $\begin{bmatrix} 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 1 & 1 & 5 \end{bmatrix}$ **in echelon form?**

The answer is NO because the third row that has only 0s (zeros) in it is placed above the fourth row which has some non zero entries. This violates the condition 1 for an echelon form.

We can address this by interchanging the third and the fourth rows (a row operation) and obtain

$$\begin{bmatrix} 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 9 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ldots\ldots\ldots(\mathbf{1})$$

It is still not in echelon form, it violates the condition number 2 because the first non zero entry of the row3 is NOT on the right of the first non zero entry of the row 2.

We may address this by interchanging the second and the third rows and obtain

$$\begin{bmatrix} 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 9 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ldots\ldots(\mathbf{2})$$

Now this matrix is in an echelon form but is still NOT in reduced echelon form because
it fails to satisfy the additional conditions namely the conditions number 4 and 5.

Fails the condition 4 because the leading entry of the row number 2 does not equal 1, it is 9,
(more detail:

$$\begin{bmatrix} 1 & 0 & & 3 & 2 & 1 & 0 \\ 0 & 9 & \text{should have 1 here} & 0 & 1 & 1 & 5 \\ 0 & 0 & & 0 & 1 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 & 0 \end{bmatrix} \ldots\ldots\ldots(\mathbf{3})$$

It also
Fails condition the number 5 because there are non zero numbers in the column of the leading entry of the third row.
(more detail: the leading entry of the third row is 1, which is good. The column of this leading entry is the fourth column. Each entry other than the leading entry has to be 0,
i.e.

$$\begin{bmatrix} 1 & 0 & 3 & 2 \text{ should have 0 here} & 1 & 0 \\ 0 & 9 & 0 & 1 \text{ should have 0 here} & 1 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ldots\ldots(\mathbf{4})$$

We may use the following row operations to transform this matrix to the row reduced echelon form.

Recall the matrix at the stage 2,

$$\begin{bmatrix} 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 9 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow \ \frac{1}{9}\mathbf{r}_2$$

$$\begin{bmatrix} 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} & \frac{1}{9} & \frac{5}{9} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{r}_1 - 2\mathbf{r}_3 \;\downarrow\; \mathbf{r}_2 - \frac{1}{9}\mathbf{r}_3$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{9} & \frac{5}{9} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we have the reduced echelon form for the given matrix.

Note that there may be different echelon forms for the same matrix but there is only
one reduced echelon form.

Please make sure that you understand the pages 14 and 15 in the text.

Please read the definition of pivot position and pivot column on the page 16.