Lesson 19:

In this lesson, we shall generalize the notion of writing the coordinate vectors of a vector with respect to different bases that you did in the section 2.9 (both editions.)

Recall the procedure by the following example

We would like to find the coordinates of $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ **relative to the basis**

B $= \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \end{bmatrix} \right\}$

If the coordinates are u and v then

$\begin{bmatrix} 1 \\ -2 \end{bmatrix}\mathbf{u}+\begin{bmatrix} 5 \\ -6 \end{bmatrix}\mathbf{v}=\begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 5 & 4 \\ -2 & -6 & 0 \end{bmatrix}$**, row echelon form:** $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 2 \end{bmatrix}$

therefore the coordinates are $u = -6, v = 2$ **or** $\begin{bmatrix} 4 \\ 0 \end{bmatrix}_B = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

NOTE THAT:

$\begin{bmatrix} 1 & 5 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} -6 \\ 2 \end{bmatrix}=\begin{bmatrix} 4 \\ 0 \end{bmatrix}$

For a given basis say $B = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \end{bmatrix} \right\}$**, the matrix formed by the columns of the basis is called the** change of coordinates matrix **from B to the standard basis and is denoted by** $\mathbf{P}_B$**.**

In this example,

$\mathbf{P}_B=\begin{bmatrix} 1 & 5 \\ -2 & -6 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 0 \end{bmatrix}_B=\begin{bmatrix} -6 \\ 2 \end{bmatrix}$

It is not hard to prove that in general, if x is a vector in terms of the standard basis, then

$\mathbf{P}_B[x]_B = \mathbf{x}$

Since the vectors of B are linearly indepedent, by the invertible matrix theorem, $\mathbf{P}_B$ is invertible, and

$P_B[x]_B = x \Rightarrow [x]_B = P_B^{-1}x$

The equation, $[x]_B = \mathbf{P}_B^{-1}\mathbf{x}$, will be very useful in future section when we transform coordinates from one non standrad basis to another non standard basis.

Recall that S=$\{1, x, x^2, \ldots, x^n\}$ is a basis of the vector space of the polynomials $\mathbf{P}_n$. We call S the standard basis of $\mathbf{P}_n$.
For example, $\{1, x, x^2\}$ is the standard basis of $\mathbf{P}_2$. Take a vector $3 - t + 2t^2$ in $\mathbf{P}_2$, its coordinate vectors with respect to the standard basis is

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

Check that the map $a_0 + a_1 t + a_2 t^2 \rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ is an isomorphism between P_2 and R^3.

To see coordinate vectors with respect to a non standard basis , let us consider the following example.
14. Page 254 Third Edition
We are given a non standard basis B=$\{1 - t^2, t - t^2, 2 - 2t + t^2\}$ of $\mathbf{P}_2$ and we have to find the coordinate vectors of
p(t)$=3 + t - 6t^2$

Note that
If u,v,w are the coordinates
$\mathbf{u}(1 - t^2) + \mathbf{v}(t - t^2) + \mathbf{w}(2 - 2t + t^2) = \mathbf{3 + t - 6t^2}$
combining like terms and equating the coefficients of like powers on each side, note that
$\mathbf{u + 2w = 3}$
$\mathbf{v - 2w = 1}$

$-\mathbf{u} - \mathbf{v} + \mathbf{w} = -6$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

therefore

$\mathbf{7}(1 - t^2) - \mathbf{3}(t - t^2) - \mathbf{2}(2 - 2t + t^2) = \mathbf{3 - 6t^2 + t}$

IMPORTANT: SEE THE CONNECTION BETWEEN THE ABOVE CALCULATIONS AND THE COORDINATES OF THE VECTORS INVOLVED.

The coordinate vectors make several calculations regarding polynomials very simple. For example if we have to check whether a given subset of polynomials is linearly independent, we can simply check whether the coordinate vectors are linearly independent.

As an example, let us consider

28. on page 255 in the third edition, in which, we have to check whether the polynomials $1 - 2t^2 - 3t^3$**,** $t + t^3$**,** $1 + 3t - 2t^2$ **form a linearly independent subset of** $\mathbf{P}_2$.

Look at the matrix whose columns are the coordinate vectors of the given polynomials.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since not all the columns are pivot columns, the above set is not linearly independent.

another good example is

32 on page 255 in the third edition, which gives us the polynomials

$\mathbf{p}_1(t) = 1 + t^2$, $\mathbf{p}_2(t) = 2 - t + 3t^2$, $\mathbf{p}_3(t) = 1 + 2t - 4t^2$ **in** $\mathbf{P}_2$.

We have to

a) To use coordinate vectors to show that these polynomials form a basis of $\mathbf{P}_2$**.**

b) To find q in P_2 such that $[q]_B = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

a)

To check the linear independence

Look at the matrix formed by using the coordinate vectors of the above vectors as columns

$$A=\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}$$

note that

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since each column is a pivot column, the columns are linearly independent and so are the polynomials.

To check the spanning part:

We have to check if any polynomial $a + bt + ct^2$ can be written as a linear combination of the above polynomials.

This is equivalent to saying that any $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in R^3 is a linear combination of the coordinate vectors of p_1, p_2, p_3

or in other words the coordinate vectors of p_1, p_2, p_3 span R^3.

Since each row of the matrix A formed by the coordinate vectors has pivot position in each row, the coordinate vectors span R^3.

Please work on the assigned exercises in the section 4.4 and post the difficulties in the discussion area.