Lesson 18:

Here we will generalize the concepts of linear independence and basis that we did before only in the context of R^n.

Recall that a set $\{v_1, v_2, \ldots, v_n\}$ of a vector space V is linearly independent iff

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 0$

A subset of V is linearly dependent if it is not linearly independent.

It can be proved that if a subset S of V is linearly dependent, we can find a vector in S which is a linear combination of the remaining vectors of S.

Also recall that a subset B of a vector space V is a basis of V iff

a) B is linearly independent.

b) B spans V.

#4 on the page 243 in the third edition, which gives us the vectors $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$ and we have to check if it is basis of R^3.

Look at the matrix $\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$ whose columns are the given vectors.

$\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

since each column is a pivot column the columns are linearly independent, therefore the subset containg the above vectors is

linearly independent.
since each row has a pivot position in it, the columns span R^3, therefore the set of the given vectors spans R^3.

This shows that $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix} \right\}$ is a basis of R^3.

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6 on page 243 in the third edition in which we are given a subset $\left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} \right\}$ of R^3 and have to

a) check if these are linearly independent.
b) if this subset spans R^3.

a) Look at $\begin{bmatrix} 1 & 1 \\ -4 & -4 \\ 3 & 3 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Since there is only one pivot column, the columns are not linearly independent.

b)
YOU MAY SEE DIRECTLY THAT THE SUBSET HAS ONLY TWO VECTORS, THEREFORE CAN NOT SPAN R^3.
or look at

$\begin{bmatrix} 1 & 1 \\ -4 & -4 \\ 3 & 3 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

not all the rows have pivot positions therefore the columns can not span R^3.

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8 on page 243 in the third edition in which we are given a

subset $\left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \right\}$ of R^3 and have to

a) check if these are linearly independent.

b) if this subset spans R^3.

a) We have more vectors than the number of entries in the vector therefore these vectors can not be linearly independent because

$\begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix}$ can not have all the columns as pivot columns

as also shown by

$\begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$

b)

But since each row in the matrix formed by these column vectors has a pivot position, therefore these columns span R^3.

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16 on page 243 in the third edition in which we are given a

subset $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} \right\}$ of R^4 and have to

find a basis for the subspace spanned by this subset.

Note that the subspace spanned by this suset is ColA where

A= $\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & 1 & -4 & 1 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

note that the columns #1,2,3,5 are the pivot columns, choose them from the original matrix, and a basis is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} \right\}$$

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32 on the page 244 in the third edition in which

we are Given that: $T(v_1), T(v_2), \ldots.., T(v_p)$ are linearly dependent and that T is one-one

Have to show that $v_1, v_2, \ldots\ldots, v_p$ are linearly dependent.

Proof:

Recall that a subset (with more than one vector) is linearly dependent iff one of the vectors can be writeen as a linear combination of the remaining vectors.

Because

$T(v_1), T(v_2), \ldots.., T(v_p)$ are linearly dependent

should be able to find j and scalar coefficients α_i such that $1 < j \leq p$

and $T(v_j) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_{j-1} T(v_{j-1}) + \alpha_{j+1} T(v_{j+1}) + \ldots. + \alpha_p T(v_p)$

gives that

$T(v_j) == T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \ldots. + \alpha_p v_p)$ because T is linear

$\rightarrow v_j = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \ldots. + \alpha_p v_p$ because T is one-one

therefore could express one of v_is as a linear combination of the remaining v_i.

Therefore $v_1, v_2, \ldots\ldots, v_p$ are linearly dependent.

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34 on the page 244 in the third edition which gives three polynomials $p_1(t) = 1 + t$, $p_2(t) = 1 - t$, $p_3(t) = 2$ (for all t)
First, we have to write a linear dependence relation among p_1, p_2, p_3 by inspection.
Second, we have to find a basis of Span$\{p_1, p_2, p_3\}$.

First:
note that $2 = (1 + t) + (1 - t)$
therefore $p_3 = p_1 + p_2$

Second Part:
Before working on this part, please the Spanning Theorem from this section (section 4.3)
Note that since $p_3 = p_1 + p_2$
Span$\{p_1, p_2, p_3\}$ = Span$\{p_1, p_2\}$
Note that p_1 , p_2 are linearly independent because they neither of them can be written in terms of the other.
therefore $\{p_1, p_2\}$ is a basis of Span$\{p_1, p_2, p_3\}$.

Please work on the exercises in the section 4.3 and post your difficulties in the discussion area.