Lesson 16:

Please note that the proof writing is going to become an important part of our study from this chapter.

The study of the rest of the chapter 4 is a generalization of the concepts learnt in the earlier chapters.

Please read the theorems and examples in the section 4.2 (both the editions) before reading this lesson.

In this lesson we shall continue working on subspaces of a vector space and linear transformations.

Example 16.1:

Let us work on #6 on the page 234 in the third edition,

we are given a matrix A= $\begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, we have to list the vectors that span the NulA.

Note that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in$NulA $\Leftrightarrow \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

it is easy to see that $x_3,\ x_4,\ x_5$ are free variable, but to get a better idea,

let us row reduce the augmented matrix further

$$\begin{bmatrix} 1 & 5 & -4 & -3 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$, row echelon form: $$\begin{bmatrix} 1 & 0 & 6 & -8 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore

$x_1 + 6x_3 - 8\ x_4 + x_5 = 0 \rightarrow x_1 = -6x_3 - 8\ x_4 - x_5$

$x_2 - 2x_3 + x_4 = 0 \rightarrow x_2 = 2x_3 - x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore NulA is spanned by $\begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Example 16.2:

Look at #8 on page 234 in the third edition, which gives us the subset $W = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 5r - 1 = s + 2t \right\}$

and we have to check whether W is a subspace of R^3.

Remember the first thing that we typically note is that whether $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$.

In this case $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin$W because $5(0) - 1 = 0 + 2(0)$ is not true which is a condition to be in W.
This reason is enough to conclude that W is not a subsdpace of R^3.

But let us note more

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \in$W because $5(1) - 1 = 2 + 2(1)$ is true

But $-\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \notin$W because $5(-1) - 1 = (-2) + 2(-1)$ is false

Example 16.3:

Look at #10 on page 234 in the third edition, which gives us the subset W$=\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a + 3b = c \\ b + c + a = d \end{matrix} \right\}$

and we have to check whether W is a subspace of R^4.

We shall work on this in two ways.

One way:

1. Note that $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in$W because $\begin{matrix} 0 + 3(0) = 0 \\ 0 + 0 + 0 = 0 \end{matrix}$

2. $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in W \Rightarrow$

$$\begin{matrix} a + 3b = c \\ b + c + a = d \end{matrix} \quad \Rightarrow \quad \begin{matrix} (-a) + 3(-b) = -c \\ (-b) + (-c) + (-a) = -d \end{matrix} \quad \Rightarrow \quad \begin{bmatrix} -a \\ -b \\ -c \\ -d \end{bmatrix} \in W$$

$$\Rightarrow \quad -\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in W$$

3. Let α be a real number.

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in W \Rightarrow$

$$\begin{matrix} a + 3b = c \\ b + c + a = d \end{matrix} \quad \Rightarrow \quad \begin{matrix} \alpha a + \alpha 3b = \alpha c \\ \alpha b + \alpha c + \alpha a = \alpha d \end{matrix} \quad \Rightarrow \quad \begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \\ \alpha d \end{bmatrix} \in W$$

$$\Rightarrow \alpha \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in W$$

Another way:
look at the conditions

$a + 3b = c \quad \rightarrow \quad a + 3b - c = 0$
$b + c + a = d \quad \rightarrow \quad a + b + c - d = 0$

row reduce the augmented matrix $\begin{bmatrix} 1 & 3 & -1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 2 & -\frac{3}{2} & 0 \\ 0 & 1 & -1 & \frac{1}{2} & 0 \end{bmatrix}$

c and d are free variables:

$a + 2c - \frac{3}{2}d = 0$

$b - c + \frac{1}{2}d = 0$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = c\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

therefore W=span$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$

Since span of a subset is a subspace, W is a subspace of R^4.

Example 16.4:

Let us take up #14 on the page 234 of the third edition, we are given W=$\left\{ \begin{bmatrix} -a + 2b \\ a - 2b \\ 3a - 6b \end{bmatrix} : \text{ a and b are real numbers} \right\}$, and we have to check whether

W is a subspace of R^3.

In this case it is easy to see that $\begin{bmatrix} -a + 2b \\ a - 2b \\ 3a - 6b \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix}$

therefore W= $\left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix} \right\}$.

Since span of a subset is a subspace, W is a subspace of R^3.

Example 16.5:

Let us take up #14 on the page 234 of the third edition, we are given A= $\begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and w= $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

To determine if w is in ColA and to determine if w is in NulA.

First let us check if w is in ColA
in order for that to happen we need to check if the augmented matrix $\begin{bmatrix} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{bmatrix}$ corresponds to a consistent system.

Note that $\begin{bmatrix} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

shows that the answer is YES (a good review exercise will be express w as a linear combination of the columns of A.)

Second let us determine if w is in NulA, for which all we need is to see if Ax=0

Note that $\begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

shows that $w \in$ NulA.

We shall continue with the section 4.2 in the Lesson 17.