Lesson 15:
From this lesson onwards, our contents will gradually become more abstract.

Must read the definition and examples of a vector space and a subspace before going ahead.

In addition to R^n , the two other vector spaces that we shall often consider are

1. C[a,b] the set of all continuous real valued functions.
2. P_n: The polymonials of degree less than or equal to n with real coefficients.

Example 15.1:

Consider #6 on the page 223 in the third edition, which asks us whether the subset consisting of the polynomials of the form $p(t)=a+t^2$, where a is in R. We have to examine whether this subset is a subspace of P_2.

Recall that in order to be a subspace a subset must have the zero vector in it.
But $a+t^2 \neq 0$ for any value of a, therefore this subset can not be a subspace.

Example 15.2:

Consider #8 on the page 223 in the third edition, which asks us to check whether the subset H containing the polynomials p(t) such that $p(0)=0$, is a subspace of P_n.

Let us check if the requirements for being a subspace (P220 third ed, P214 in the second updated ed) are met.

Note that the zero polynomial is 0 everywhere, therefore 0 at 0 as well, therefore belongs to H.
If p and q are in H, p(0)=0,q(0)=0, and (p+q)(0)=p(0)+q(0)=0 as well, therefore p+q is in H.
If p is in H, p(0)=0, therefore for any real number c, (cp)(0)=cp(0)=c0=0, therefore cp is in H.

Therefore H is a subspace of P_n.

Example 15.3:

Consider #12 on the page 223 in the third edition, which asks us to show that W the set of all vectors of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$ is a subspace of R^4.

Note that $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$

therefore W=Span$\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}\right\}$,

Since span of a subset is a subspace, W is a subspace of R^4.

Example 15.4:

Consider #16 on the page 223 in the third edition, which asks us to check if W the set of all vectors of the form $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$ is a subspace of R^3.

In this case, because of the top entry, we can not write W as a linear combination in a way we did in the last example.

But note that in order to be a subspace, W must have the zero vector, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in W.

But if $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then

$-a+1 = 0$
$a-6b = 0$
$2b+a = 0$
or
$a = 1$
$a-6b = 0$
$a+2b = 0$

the augmented matrix

$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -6 & 0 \\ 1 & 2 & 0 \end{bmatrix}$, with the row echelon form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ shows that the above system is inconsistent, and therefore $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, and W is not a subspace of R^3.

alternatively:

we can note that

$a = 1$
$a - 6b = 0 \rightarrow 1 - 6b = 0 \rightarrow b = \frac{1}{6}$
$a + 2b = 0$ but $1 + 2\left(\frac{1}{6}\right) = \frac{4}{3} \neq 0$ which is a contradiction.

Example 15.5:

Consider #32 on the page 225 in the third edition.

First Part:
Given that H and K are subspaces of a vector space V.
To show that $H \cap K$ is a subspace of V.

Proof:

Since H and K are subspaces, $0 \in H$ and $0 \in K$, therefore $0 \in H \cap K$

$u, v \in H \cap K$,
$\Rightarrow u, v \in H$ and $u, v \in K$
$\Rightarrow u+v \in H$ and $u+v \in K$ because H and K are subspaces
$\Rightarrow u+v \in H \cap K$

$c \in R$ and $x \in H \cap K$
$\Rightarrow c \in R$ and $x \in H$ and $x \in K$
$\Rightarrow cx \in H$ and $cx \in H$ because H and K are subspaces
$\Rightarrow cx \in H \cap K$

therefore $H \cap K$ is a subspace of V.

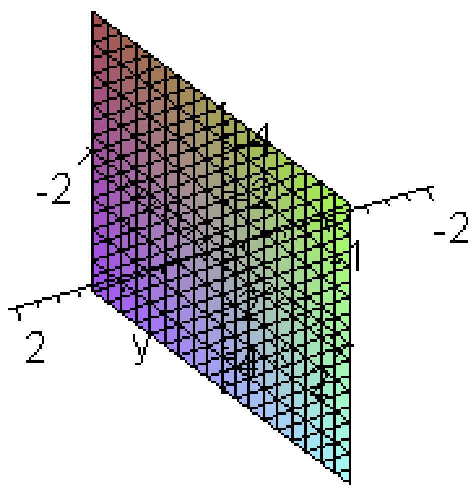
Second Part:

We have to give examples of subspaces H and K of R^2 such that $H \cup K$ is not a subspace.

Let us work this in R^3 and leave the examples of R^2 as an exercise.

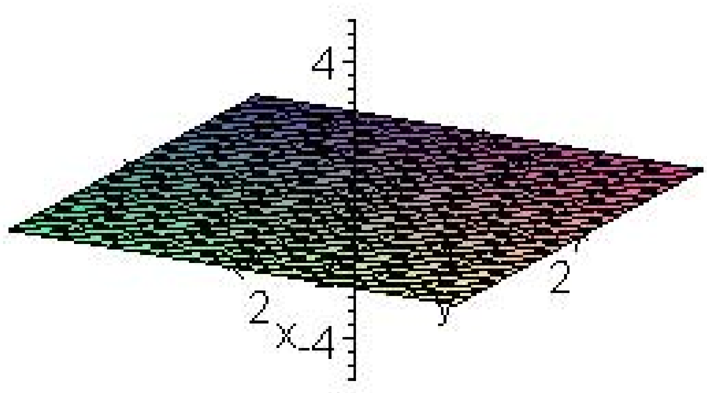
Consider $H=\left\{\begin{bmatrix} 0 \\ y \\ z \end{bmatrix} : \text{ y and z any real numbers}\right\} \subset R^3$

we can easily see that H is a subspace of R^3.



Consider $K=\left\{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : \text{ x and y any real numbers}\right\} \subset R^3$

we can easily see that K is a subspace of R^3.



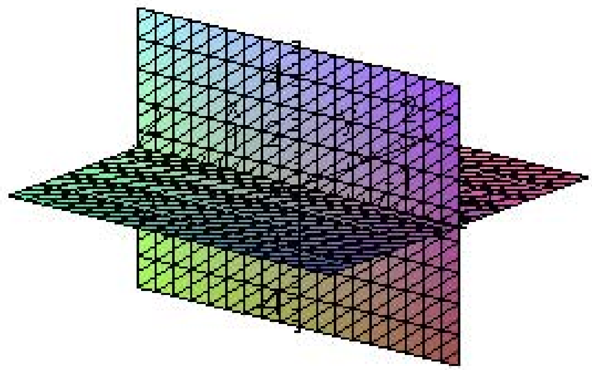
But H∪K (the union of yz plane and the xyplane) is not a subspace because

even though $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H \cup K$

but

$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in H$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in K$ but $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \notin H \cup K$

a geometric view of H∪K is



Please work on the 4.1 exercises and post your questions in the discussion.