The Ultimate Guess

Please read the pages 736-742 in the text book

Under the permissible conditions, we are going to assume that an infinite series has a solution in the form of a power series and then shall try to determine the solution by substitution in the differential equation.

The approach is analogous to the approach that we had by expanding functions in infinite series and then using the solution to approximate integrals like $\int_0^{0.3} \cos x^2 dx$

Example 1:

#4 on the page 742

To find the solution of the differential equation $\frac{dy}{dt} = t^2 y + 1$

First, note that if we use our earliar methods, the resulting integral will not be computable

therefore let us assume that the solution of this differential equation is

$$y = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \ldots\ldots + a_{n-1} t^{n-1} + a_n t^n + \ldots\ldots\ldots$$

This means that

$$\frac{dy}{dt} = a_1 + 2a_2 t + \ldots\ldots + (n-1)a_{n-1} t^{n-2} + n a_n t^{n-1} + \ldots\ldots\ldots$$

Substituting this in the differential equation

$$\frac{dy}{dt} = t^2 y + 1$$

$$a_1 + 2a_2 t + \ldots\ldots + (n-1)a_{n-1} t^{n-2} + n a_n t^{n-1} + \ldots\ldots\ldots = t^2(a_0 + a_1 t + a_2 t^2 + \ldots\ldots + a_{n-1} t^{n-1} + a_n t^n + \ldots\ldots\ldots) + 1$$

or

$$a_1 + 2a_2 t + \ldots\ldots + (n-1)a_{n-1} t^{n-2} + n a_n t^{n-1} + \ldots\ldots\ldots = 1 + t^2(a_0 + a_1 t + a_2 t^2 + \ldots\ldots + a_{n-1} t^{n-1} + a_n t^n + \ldots\ldots\ldots)$$

Since, we have to give an approximation of only upto the degree 4,
we shall not stop the expansion after the fourth degree term

$$a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \ldots\ldots\ldots = 1 + t^2(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots\ldots\ldots\ldots)$$

$$a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + \ldots\ldots\ldots = 1 + a_0t^2 + a_1t^3 + a_2t^4 + a_3t^5 + \ldots\ldots\ldots\ldots$$

$$a_1 - 1 + 2a_2t + (3a_3 - a_0)t^2 + (4a_4 - a_1)t^3 + (5a_5 - a_2)t^4 + \ldots\ldots\ldots = 0$$

$$(a_1 - 1) + 2a_2t + (3a_3 - a_0)t^2 + (4a_4 - a_1)t^3 + (5a_5 - a_2)t^4 + \ldots\ldots\ldots = 0$$

Setting each of the coefficients on the left hand side to 0

we have

$$a_1 = 1$$
$$a_2 = 0$$
$$a_3 = \tfrac{1}{3}a_0$$
$$a_4 = \tfrac{1}{4}a_1 = \tfrac{1}{4}$$

the power series solution upto degree 4 is

$$y = a_0 + t + \tfrac{1}{3}a_0t^3 + \tfrac{1}{4}a^4 + \ldots\ldots$$

we may treat a_0 as a parameter that is any subsitution for a_0 will give us a solution

Example 2:

#12 on the page 742

To find the general solution upto degree 6 for

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + t^2 y = \cos t$$

If

$$y = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \ldots\ldots + a_{n-1} t^{n-1} + a_n t^n + \ldots\ldots\ldots$$

is a solution

$$\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + 6a_6 t^5 + 7a_7 t^6 + 8a_8 t^7 + \ldots.$$

$$\frac{d^2y}{dt^2} = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 30a_6 t^4 + 42a_7 t^5 + 56a_8 t^6 + \ldots\ldots$$

(I am multiplying the coefficients out but in theoretical situations, it is a good idea to retain the factored form to catch a pattern)

Recall that McLaurin's series of $\cos t$ is

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots..$$

Substitution in the differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + t^2 y = \cos t$$

gives

$$(2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 30a_6 t^4 + 42a_7 t^5 + 56a_8 t^6 + \ldots..)$$

$$+(a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + 6a_6 t^5 + 7a_7 t^6 + \ldots)$$
$$+t^2(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \ldots.)$$
$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots..\right)$$

or

$$(2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 30a_6 t^4 + 42a_7 t^5 + 56a_8 t^6 + \ldots..)$$
$$+(a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + 6a_6 t^5 + 7a_7 t^6 + \ldots)$$
$$+(a_0 t^2 + a_1 t^3 + a_2 t^4 + a_3 t^5 + a_4 t^6 + a_5 t^7 + a_6 t^8 + \ldots.)$$
$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots..\right)$$

that is

$(2a_2 + a_1) + (6a_3 + 2a_2 + a_0)t + (12a_4 + 3a_3 + a_0)t^2 + (20a_5 + 4a_4 + a_1)t^3 + (30a_6 + 5a_5 + a_2)t^4 + (42a_7 + 6a_6 + a_3)t^5 + (56a_8$
$=1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots..$

Equating the coefficients of like powers of t, we have
$2a_2 + a_1 = 1$

$6a_3 + 2a_2 = 0$
$12a_4 + 3a_3 + a_0 = -\frac{1}{2!}$
$20a_5 + 4a_4 + a_1 = 0$
$30a_6 + 5a_5 + a_2 = \frac{1}{4!}$
$42a_7 + 6a_6 + a_3 = 0$
$56a_8 + 7a_7 + a_4 = -\frac{1}{6!}$
.....

working these out patiently with algebra

$2a_2 + a_1 = 1 \to a_2 = \frac{1}{2} - \frac{1}{2}a_1$

$6a_3 + 2a_2 = 0 \to a_3 = -\frac{1}{3}a_2 = -\frac{1}{3}(\frac{1}{2} - \frac{1}{2}a_1) = -\frac{1}{6} + \frac{1}{6}a_1$
$\to a_3 = -\frac{1}{6} + \frac{1}{6}a_1$

$12a_4 + 3a_3 + a_0 = -\frac{1}{2!} \to 12a_4 = -\frac{1}{2} - 3a_3 - a_0 = -\frac{1}{2} - 3\left(-\frac{1}{6} + \frac{1}{6}a_1\right) - a_0 = -\frac{1}{2} + \frac{1}{2} - \frac{1}{2}a_1 - a_0 = -\frac{1}{2}a_1 - a_0$
$\to 12a_4 = -\frac{1}{2}a_1 - a_0 \to a_4 = -\frac{1}{24}(2a_0 + a_1)$

$20a_5 + 4a_4 + a_1 = 0 \to 20a_5 = -4a_4 - a_1 = -4(-\frac{1}{24}(2a_0 + a_1)) - a_1 = \frac{1}{3}a_0 - \frac{5}{6}a_1$

$20a_5 = \frac{1}{3}a_0 - \frac{5}{6}a_1 \to a_5 = \frac{1}{60}a_0 - \frac{1}{24}a_1$

$30a_6 + 5a_5 + a_2 = \frac{1}{4!} \to 30a_6 = \frac{1}{4!} - 5a_5 - a_2 \to a_6 = \frac{1}{30}\left[\frac{1}{24} - 5\left(\frac{1}{60}a_0 - \frac{1}{24}a_1\right) - (\frac{1}{2} - \frac{1}{2}a_1)\right]$

$\to a_6 = \frac{1}{30}[\frac{1}{24} - \frac{1}{12}a_0 + \frac{5}{24}a_1 - \frac{1}{2} + \frac{1}{2}a_1]$

$\to a_6 = \frac{1}{30}[-\frac{1}{12}a_0 + \frac{17}{24}a_1 - \frac{11}{24}]$
$\to a_6 = -\frac{1}{360}a_0 + \frac{17}{720}a_1 - \frac{11}{720}$

Note that we have expressed all the coefficients in trems of a_0 and a_1 only

we stop here because the question wanted us to go only upto t^6

Therefore the solution

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \ldots\ldots$$

may be written as

$$y = a_0 + a_1 t + (\frac{1}{2} - \frac{1}{2}a_1)t^2 + \left(-\frac{1}{6} + \frac{1}{6}a_1\right)t^3 - \frac{1}{24}(2a_0 + a_1)t^4 + \left(\frac{1}{60}a_0 - \frac{1}{24}a_1\right)t^5 + \left(-\frac{1}{360}a_0 + \frac{17}{720}a_1 - \frac{11}{720}\right)t^6 + \ldots.$$

Suggested Practice

Appendix B (Page 742)

3,9,11,13