If f is a function on $[0,\infty)$, the function $\mathcal{L}(f)$defined by the integral

$$\mathcal{L}[f] = \int_0^{\infty} e^{-st} f(t)\, dt$$

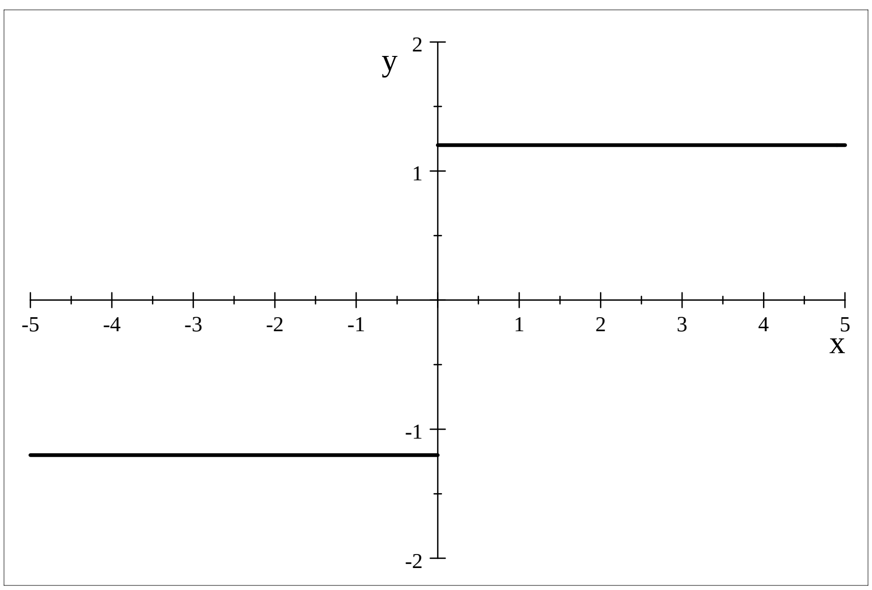
for those values of s for which the integral converges is the Laplace transform of f. The Laplace transform depends on the function f and the number s.

In our course, Laplace Transforms will be used to solve differential equations of the form,

$$\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + Q(t)y = f(t)$$

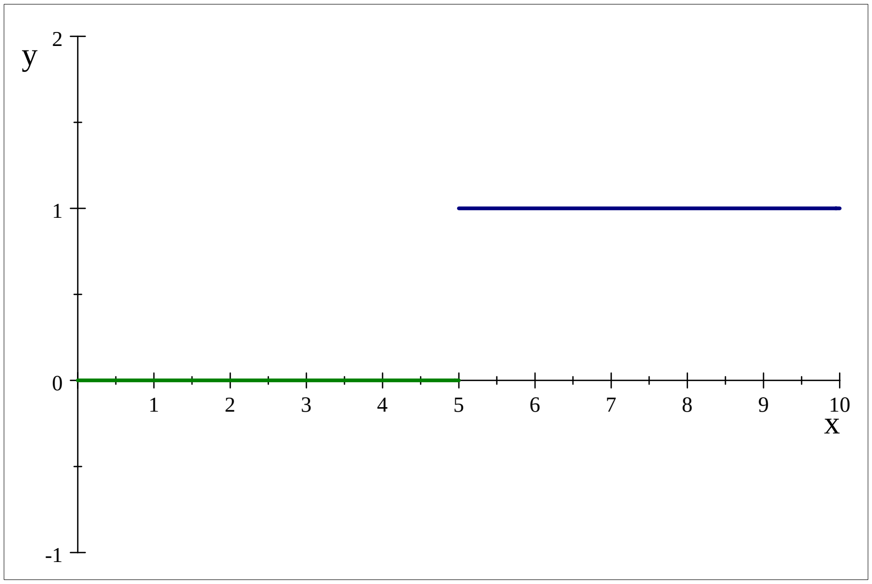
when $f(t)$ is a piece wise continuous function.

Remember, a piecewise continuous function looks like

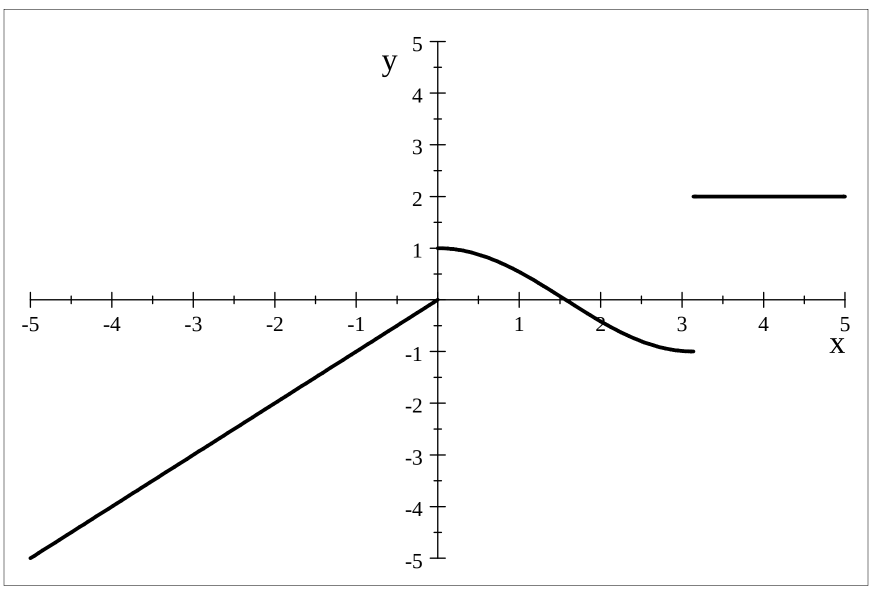


OR

0



even



Before applying the Laplace Transforms to the solutions of differential equations, let us first compute the Laplace Tansforms of some basic function.

Let α be a constant

$$\pounds[\alpha] = \int_0^\infty \alpha e^{-st}dt = \alpha\left(-\frac{e^{-st}}{s}\Big|_0^{\nearrow\infty}\right)$$

Note that

$$-\frac{e^{-st}}{s}\Big|_0^{\nearrow\infty} = Lim_{b\to\infty}\left(-\frac{e^{-sb}}{s}\right) - \left(-\frac{e^{-s(0)}}{s}\right)$$
$$-\frac{e^{-st}}{s}\Big|_0^{\nearrow\infty} = Lim_{b\to\infty}\left(-\frac{e^{-sb}}{s}\right) - \left(-\frac{1}{s}\right)$$
$$-\frac{e^{-st}}{s}\Big|_0^{\nearrow\infty} = Lim_{b\to\infty}\left(-\frac{e^{-sb}}{s}\right) + \frac{1}{s}$$

$Lim_{t\to\infty}\left(-\frac{e^{-st}}{s}\right) = 0$ if $s > 0$

$$\pounds[\alpha] = \int_0^\infty \alpha e^{-st}dt = \alpha\left(-\frac{e^{-st}}{s}\Big|_0^{\nearrow\infty}\right) = \frac{\alpha}{s}\ ,\ s > 0$$

in particular

$\pounds[1] = \frac{1}{s}$

you may check by induction that

$\boxed{\pounds[t^n] = \frac{n!}{s^{n+1}} \quad \text{for a positive integer } n}$

$$\pounds[e^{\alpha t}] = \int_0^\infty e^{-st}e^{\alpha t}dt = \int_0^\infty e^{(\alpha-s)t}dt = \frac{e^{(\alpha-s)t}}{(\alpha-s)}\Big|_0^{\nearrow\infty}$$

Note that

$-\frac{e^{(\alpha-s)t}}{(\alpha-s)}\Big|_0^{\nearrow\infty} = -\frac{1}{\alpha-s}$ if $s > \alpha$

$$\boxed{\pounds[e^{\alpha t}] = \frac{1}{s-\alpha}}$$

A sufficient condition for existence of Laplace Transform of a Piecewise continuous f function on $[0,\infty)$ is

If we can obtain numbers c , $M > 0$ and $T > 0$ such that
$|f(t)| \leq Me^{ct}$, for $t > T$

then $£[f(t)] = \int_0^\infty e^{-st}f(t)\,dt$ exists for $s > c$

Note that the Laplace transforms are linear.

Under the applicable conditions:
A Laplace Transform will transform a differential equation into an algebraic equation, which is relatively easy to solve.

We saw above examples of Laplace Transforms of functions,

let us see how does a Laplace Transform of a derivative work.

If the laplace transform $£[y]$ of a function y of t exists,

then $£\left[\frac{dy}{dt}\right] = \int_0^\infty e^{-st}\frac{dy}{dt}dt$

recall the method of the integration by parts
$\int udv = uv - \int vdu$
and note that

if we took
$u = e^{-st}, dv = \frac{dy}{dt}dt$
$du = -se^{st}dt \quad , \quad v = y$

$$\int e^{-st}\frac{dy}{dt}dt = -e^{-st}y + \int yse^{-st}dt$$
$$\int_0^\infty e^{-st}\frac{dy}{dt}dt = -e^{-st}y\Big|_0^{\nearrow\infty} + \int_0^\infty yse^{-st}dt$$

$$|y| \le Me^{ct}$$
$$s > 0$$

$$-e^{-st}y\Big|_0^{\nearrow\infty} = -Lim_{b\to\infty}(ye^{-sb}) + y(0)e^{-s(0)} = 0 + y(0) = y(0)$$

$$\int_0^\infty yse^{-st}dt = s\int_0^\infty ye^{-st}dt = s£[y]$$

therefore

$$£\left[\frac{dy}{dt}\right] = \int_0^\infty e^{-st}\frac{dy}{dt}dt = -y(0) + s£[y]$$

or

$$\boxed{£\left[\frac{dy}{dt}\right] = -y(0) + s£[y]}$$

Since, a Laplace transform will transform an a differential equation into an albegaric equation, we should have an inverse transform to get back to the original question.

$$£[f] = F \qquad \Leftrightarrow \qquad £^{-1}(F) = f$$

recalling the results that we have already obtained

$£[1] = \frac{1}{s}$ gives $£^{-1}\left[\frac{1}{s}\right] = 1$

$£[t^n] = \frac{n!}{s^{n+1}}$ gives $£^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$ for a positive integer n

$£[e^{\alpha t}] = \frac{1}{s-\alpha}$ gives $£^{-1}\left[\frac{1}{s-\alpha}\right] = e^{\alpha t}$

Note that the inverse Laplace Transform is also linear.

Example of the computation of the inverse transform (Laplace in the case of this lesson)

To find

$$£^{-1}\left[\frac{2s+4}{(s-2)(s^2+4s+3)}\right] = £^{-1}\left[\frac{2s+4}{(s-2)(s+1)(s+3)}\right]$$

We know from the results shown above that

$£^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$ $\boxed{\text{Put } \alpha = 2 \text{ in } £^{-1}\left[\frac{1}{s-\alpha}\right] = e^{\alpha t}}$

$£^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$ $\boxed{\text{Put } \alpha = 2 \text{ in } £^{-1}\left[\frac{1}{s-\alpha}\right] = e^{\alpha t}}$

$£^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$ $\boxed{\text{Put } \alpha = 3 \text{ in } £^{-1}\left[\frac{1}{s-\alpha}\right] = e^{\alpha t}}$

We can use these values if we can express

$\frac{2s+4}{(s-2)(s+1)(s+3)}$ as a sum of fractions in the manners hown below by using the

methods of Partial Fractions

$$\frac{2s+4}{(s-2)(s+1)(s+3)}$$
$$= \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{s+3}$$
$$= \frac{A(s+1)(s+3) + B(s-2)(s+3) + C(s-2)(s+1)}{(s-2)(s+1)(s+3)}$$

$$= \frac{A(s^2+4s+3)+B(s^2+s-6)+C(s^2-s-2)}{(s-2)(s+1)(s+3)}$$
$$= \frac{As^2+4As+3A+Bs^2+Bs-6B+Cs^2-Cs-2C}{(s-2)(s+1)(s+3)}$$
$$= \frac{(A+B+C)s^2+(4A+B-C)s+(3A-6B-2C)}{(s-2)(s+1)(s+3)}$$

compare the two sides of

$$\frac{2s+4}{(s-2)(s+1)(s+3)} = \frac{(A+B+C)s^2+(4A+B-C)s+(3A-6B-2C)}{(s-2)(s+1)(s+3)}$$

and note that

$A+B+C = 0$ .... (1)
$4A+B-C = 2$ .... (2)
$3A-6B-2C = 4$ ...... (3)

Let us recall several methods from AlgebraII to solve the above system

Add equation (1) to equation (2) to get $5A+2B = 2$ ... (4)
Add 2 times equ (1) to equ (3) to get $5A-4B = 4$ ... (5)

subtract equation (5) from the equation (4)

$$6B = -2 \to B = -\frac{1}{3}$$

equation (4) with the substitution of the above value gives

$$5A+2\left(-\frac{1}{3}\right) = 2$$
$$5A-\frac{2}{3} = 2$$
$$5A = 2+\frac{2}{3}$$

$$5A = \frac{8}{3}$$
$$A = \frac{8}{15}$$

equation (1) with substitution of
$$A = \frac{8}{15}, B = -\frac{1}{3}$$

gives
$$A + B + C = 0$$
$$\frac{8}{15} - \frac{1}{3} + C = 0$$
$$\frac{8}{15} - \frac{5}{15} + C = 0$$
$$\frac{3}{15} + C = 0$$
$$C = -\frac{1}{5}$$

therefore

$$\frac{2s+4}{(s-2)(s+1)(s+3)} = \frac{8}{15(s-2)} - \frac{1}{3(s+1)} - \frac{1}{5(s+3)}$$

$$£^{-1}\left[\frac{2s+4}{(s-2)(s+1)(s+3)}\right] = \frac{8}{15}£^{-1}\left[\frac{1}{s-2}\right] - \frac{1}{3}£^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{5}£^{-1}\left[\frac{1}{s+3}\right]$$
$$£^{-1}\left[\frac{2s+4}{(s-2)(s+1)(s+3)}\right] = \frac{8}{15}e^{2t} - \frac{1}{3}e^{-t} - \frac{1}{5}e^{-3t}$$

you may use method of elimination to solve these or if you are using a calculator like

TI83

You may write the

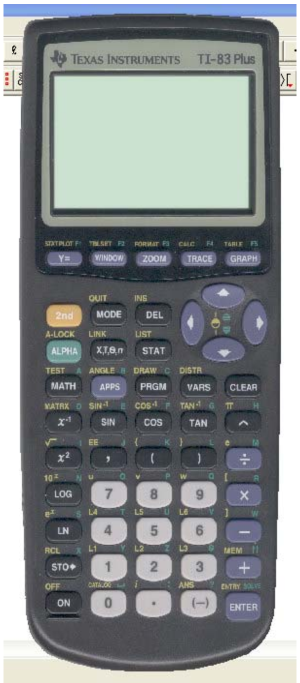
system of equations
$$A + B + C = 0$$
$$4A + B - C = 2$$
$$3A - 6B - 2C = 4$$
in the matrix form

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 4 & 1 & -1 & 2 \\ 3 & -6 & -2 & 4 \end{bmatrix}$$

to obtain the row reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & \frac{8}{15} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{5} \end{bmatrix}$$

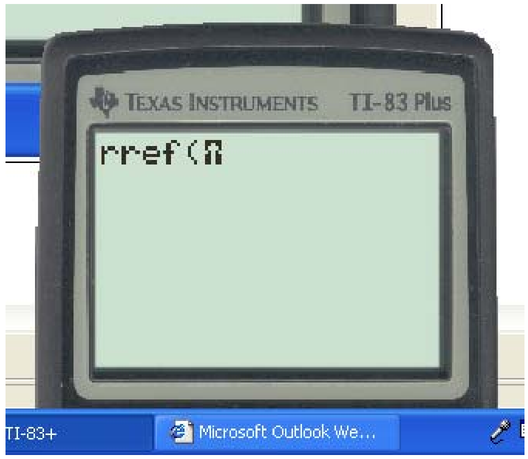
by using



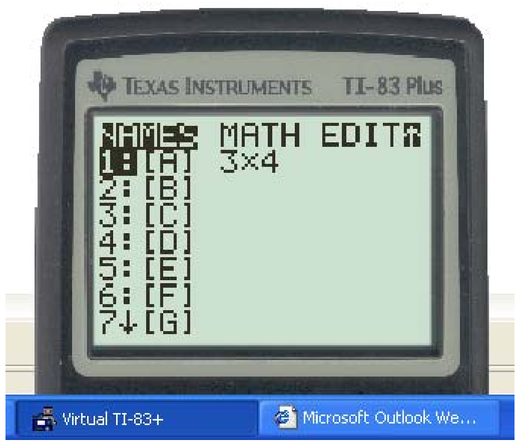
the MATRIX option

Edit the matrix A to suit enter our matrix in the following way

MAKE SURE TO QUIT
go the matrix option, select MATH, and chose rref (row reduced echelon form)

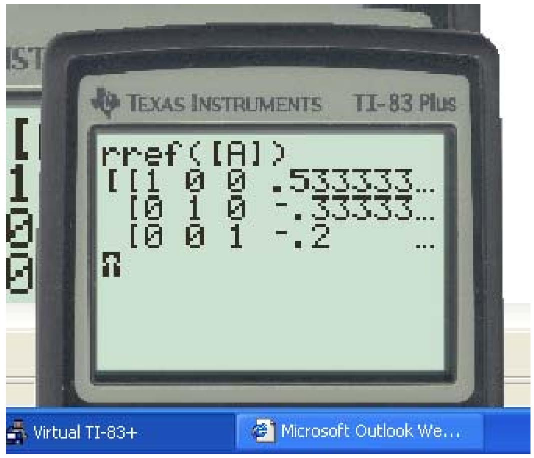


Go back to matrix and press enter when you have our matrix A



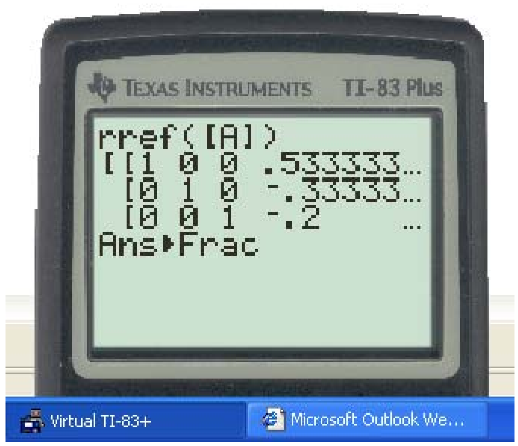
you should see

press ENTER
to obatin

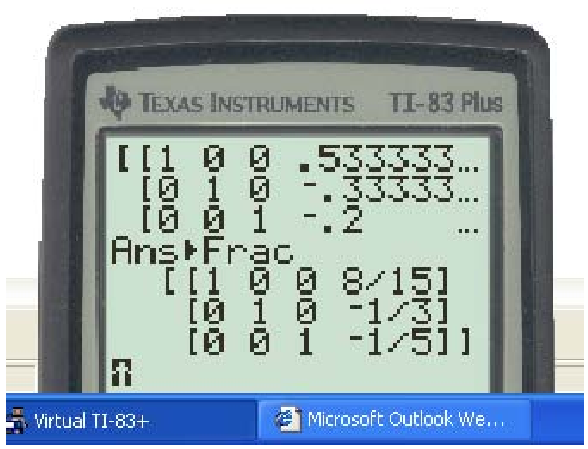


If you would like the answers in the fraction form:

press MATH then Frac to obtain



ENTER



Let us look at the following example of using a Laplace Transform to solve an initial value problem.

(#24 on the page 572 of the text book)

$$\frac{dy}{dt} + 4y = 2 + 3t, \quad y(0) = 0$$

Even though the differential equation can be solved without Laplace Transforms, let us use it to learn the procedure.

$$\frac{dy}{dt} + 4y = 2 + 3t$$

Because the Linearity of the Laplace Transforms

$$£\left[\frac{dy}{dt}\right] + 4£[y] = 2£[1] + 3£[t]$$

Use $£[1] = \frac{1}{s}$, $£[t^n] = \frac{n!}{s^{n+1}}$ and $£\left[\frac{dy}{dt}\right] = -y(0) + s£[y]$

$$-y(0) + s£[y] + 4£[y] = \frac{2}{s} + \frac{3}{s^2}$$

Given that $y(0) = 1$

$$-1 + s£[y] + 4£[y] = \frac{2}{s} + \frac{3}{s^2}$$

$$(s+4)£[y] = \frac{2}{s} + \frac{3}{s^2} + 1$$

$$£[y] = \frac{2}{s(s+4)} + \frac{3}{s^2(s+4)} + \frac{1}{s+4}$$
$$£[y] = \frac{s^2+2s+3}{s^2(s+4)}$$

therefore

$$y = £^{-1}\left[\frac{s^2+2s+3}{s^2(s+4)}\right]$$

Use the method of Partial Fractions

$$\frac{s^2+2s+3}{s^2(s+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+4}$$

$$\frac{s^2+2s+3}{s^2(s+4)} = \frac{As(s+4)+B(s+4)+Cs^2}{s^2(s+4)}$$
$$\frac{s^2+2s+3}{s^2(s+4)} = \frac{As^2+4As+Bs+4B+Cs^2}{s^2(s+4)}$$
$$\frac{s^2+2s+3}{s^2(s+4)} = \frac{(A+C)s^2+(4A+B)s+4B}{s^2(s+4)}$$

$A + C = 1$
$4A + B = 2$
$4B = 3 \to B = \frac{3}{4}$
$4A + B = 2 \to 4A + \frac{3}{4} = 2 \to 4A = 2 - \frac{3}{4} \to 4A = \frac{5}{4} \to A = \frac{5}{16}$
$A + C = 1 \to C = 1 - A \to C = 1 - \frac{5}{16} = \frac{11}{16}$

$$\frac{s^2+2s+3}{s^2(s+4)} = \frac{5}{16s} + \frac{3}{4s^2} + \frac{11}{16(s+4)}$$

$$y = £^{-1}\left[\frac{s^2+2s+3}{s^2(s+4)}\right] = \frac{5}{16}£^{-1}\left[\frac{1}{s}\right] + \frac{3}{4}£^{-1}\left[\frac{1}{s^2}\right] + \frac{11}{16}£^{-1}\left[\frac{1}{s+4}\right]$$

$$y = \frac{5}{16} + \frac{3}{4}t + \frac{11}{16}e^{-4t}$$

Please finsih the practice problems for the section 6.1

Laplace Transforms of Discontinuous Functions:

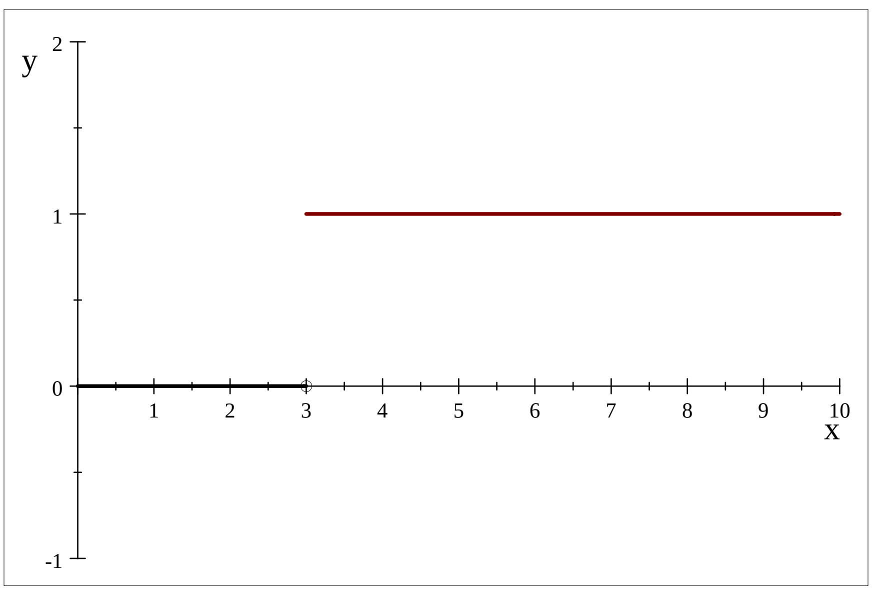
Let us consider the function

$$\mathbf{u}_a(t) = \begin{cases} 0 & if \quad t < a \\ 1 & if \quad t \geq a \end{cases}$$

For example,

$$\mathbf{u}_3(t) = \begin{cases} 0 & if \quad t < 3 \\ 1 & if \quad t \geq 3 \end{cases}$$

a graph is shown below



This function is called the Heaviside function and are used in many applications such as electrical systems.

For $a > 0$

let us compute $\pounds[u_a(t)] = \int_0^{\infty} u_a(t)e^{-st}dt$

recall that

$$u_a(t) = \begin{cases} 0 & if \quad t < a \\ 1 & if \quad t \geq a \end{cases}$$

Since a **is a crucial point in the definition**, **let us split** $\pounds(u_a(t))$ **in the following manner**

$$\pounds[u_a(t)] = \int_0^{\infty} u_a(t)e^{-st}dt = \int_0^{a} u_a(t)e^{-st}dt + \int_a^{\infty} u_a(t)e^{-st}dt$$

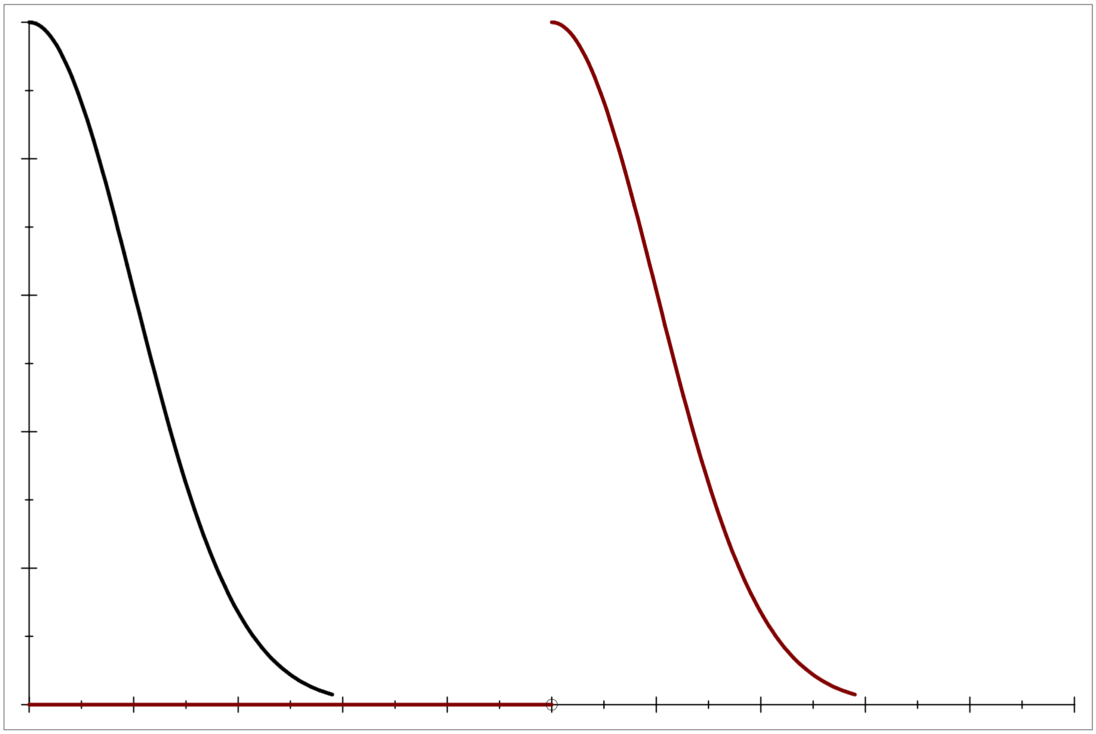
$$\pounds[u_a(t)] = \int_0^{a} (0)e^{-st}dt + \int_a^{\infty} (1)e^{-st}dt$$

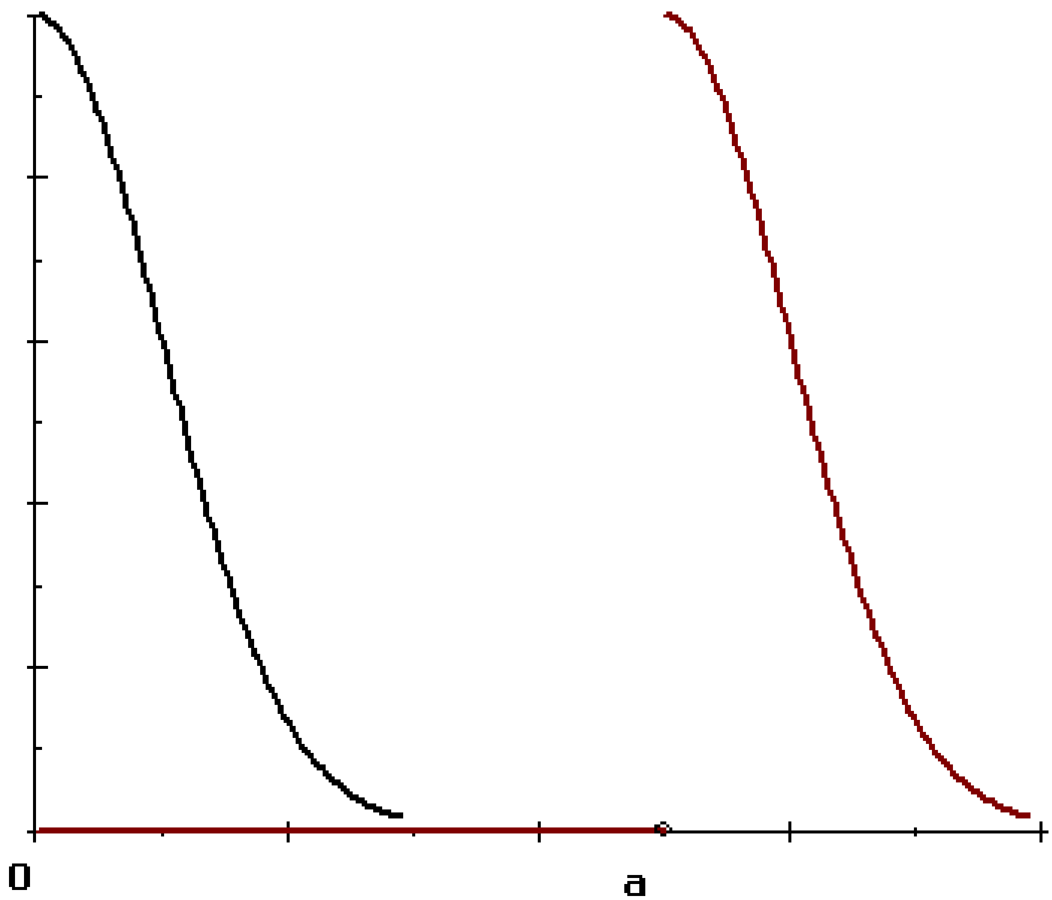
$$\pounds[u_a(t)] = \int_a^{\infty} e^{-st}dt = -\frac{e^{-st}}{s}\Big|_a^{\nearrow\infty} = \frac{e^{-sa}}{s}$$

that is

$$\boxed{\pounds[u_a(t)] = \frac{e^{-sa}}{s}}$$

Compare the functions shown below




0
a

Let the black graph shown above be a graph of f **and the red graph be a graph of** g

Notice that for $t \geq a$, **the graph of** g **is the graph of** f **shifted to the right by** a

that is to say that for $t \geq a$, $\quad g(t) = f(t-a)$

OR

$$g(t) = \begin{cases} 0 & \text{if } \ t < a \\ f(t-a) & \text{if } \ t \geq a \end{cases}$$

OR we may write

$\mathbf{g}(t) = \mathbf{u}_a(t)\mathbf{f}(t-a)$

such a transformation means that we turn the function on when $t = a$

The Laplace Transforms of g **and** f **are related in the following manner**

$$£[g] = \int_0^\infty g(t)e^{-st}dt = \int_a^\infty f(t-a)e^{-st}dt \text{ **because** } g(t) = \begin{cases} 0 & \text{if } \ t < a \\ f(t-a) & \text{if } \ t \geq a \end{cases}$$

To evaluate

$$\int_a^\infty \mathbf{f}(t-a)\mathbf{e}^{-st}\mathbf{dt}$$

take $t - a = u \rightarrow dt = du$
and also $t = u + a$

$$\int_a^{\infty} \mathbf{f}(t-a)\mathbf{e}^{-st}\mathbf{dt}$$

$$=\int_0^{\infty} f(u)e^{-s(u+a)}du$$

$$=\int_0^{\infty} f(u)e^{-su}e^{-sa}du$$

$$=e^{-sa}\int_0^{\infty} f(u)e^{-su}du$$

$=e^{-sa}F(s)$ **where** $£[f] = F(s)$

That is

$$\boxed{£[u_a(t)f(t-a)] = e^{-sa}F(s), \text{ where } £[f] = F(s)}$$

OR

$$\boxed{£^{-1}[e^{-sa}F(s)] = u_a(t)f(t-a) \text{ , where } £^{-1}[F(s)] = f(t)}$$

Before we apply this technique for the solutions of differential equations,

let us work with couple of plain examples.

Example 1:

Remember that $£[t^n] = \frac{n!}{s^{n+1}}$, **where** n **is a whole number**

Therefore $£\left[u_2(t)(t-2)^3\right] = e^{-2s}£[t^3] = e^{-2s}\left(\frac{3!}{s^4}\right) = \frac{6e^{-2s}}{s^4}$

OR

$£^{-1}\left[\frac{6e^{-2s}}{s^4}\right] = \mathbf{u}_2(t)(t-2)^3$

Example 2:

To find $\pounds^{-1}\left[\frac{4e^{-2s}}{s(s+3)}\right]$ (**#6 on the page 580 of the text**)

First, **note that** $\frac{1}{s(s+3)} = \frac{1}{3s} - \frac{1}{3(s+3)}$ (**used the methods of Partial Fractions**)

and that $\pounds^{-1}[\frac{1}{s}] = 1$
because $\pounds[t^n] = \frac{n!}{s^{n+1}}$, **where** n **is a whole number**, **therefore**

$\pounds[1]= \pounds[t^0]=\frac{0!}{s^{0+1}}=\frac{1}{s}\rightarrow \pounds^{-1}[\frac{1}{s}]= \mathbf{1}$

and also that $\pounds^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$ **BECAUSE** $\pounds[e^{at}] = \frac{1}{s-a}$

Therefore

$\pounds^{-1}\left[\frac{4e^{-2s}}{s(s+3)}\right]$
$= \pounds^{-1}\left[\frac{4e^{-2s}}{s(s+3)}\right]$
$= \pounds^{-1}\left[4e^{-2s}\left(\frac{1}{3s} - \frac{1}{3(s+3)}\right)\right]$ **Recall from the above that** $\frac{1}{s(s+3)} = \frac{1}{3s} - \frac{1}{3(s+3)}$
$=\frac{4}{3}\pounds^{-1}\left[\frac{e^{-2s}}{s}\right]- \frac{4}{3}\pounds^{-1}\left[\frac{e^{-2s}}{s+3}\right]$
$=\frac{4}{3}u_2(t) - \frac{4}{3}u_2(t)e^{-3(t-2)}$ **because**
$\boxed{\pounds^{-1}[e^{-sa}F(s)] = u_a(t)f(t-a) \text{, where } \pounds^{-1}[F(s)] = f(t)}$
$=\frac{4}{3}u_2(t)[1 - e^{-3(t-2)}]$

Applications to the solutions of initial value problems:

#12 on the page 580

$\frac{dy}{dt} = -y + 2u_3(t)$ **with** $y(0) = 3$

We may rewrite the equation:

$\frac{dy}{dt} + \mathbf{y} = \mathbf{2u}_3(t)$

apply the Laplace Transform:

$£\left[\frac{dy}{dt}\right] + £[y] = \mathbf{2}£[u_3(t)]$

recall that

$\boxed{£\left[\frac{dy}{dt}\right] = -y(0) + s£[y]}$ **and that** $\boxed{£[u_a(t)] = \frac{e^{-sa}}{s}}$

$£\left[\frac{dy}{dt}\right] + £[y] = \mathbf{2}£[u_3(t)]$

$\rightarrow$

$-\mathbf{y}(0) + \mathbf{s}£[y] + £[y] = \mathbf{2}\left(\frac{e^{-3s}}{s}\right)$

since it is given that $y(0) = 3$

We have

$-\mathbf{4} + \mathbf{s}£[y] + £[y] = \mathbf{2}\left(\frac{e^{-3s}}{s}\right)$

$\rightarrow$

$\mathbf{s}£[y] + £[y] = \mathbf{2}\left(\frac{e^{-3s}}{s}\right) + \mathbf{4}$

$\rightarrow$

$(s+1)£[y] = \mathbf{2}\left(\frac{e^{-3s}}{s}\right) + \mathbf{4}$

$\rightarrow$

$£[y] = \frac{2e^{-3s}}{s(s+1)} + \frac{4}{s+1}$

Use the method of partial fractions to obtain

$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

therefore we have

$$£[y]=\frac{2e^{-3s}}{s(s+1)}+\frac{4}{s+1}$$
$\rightarrow$

$$£[y]= \mathbf{2e}^{-3s}\left(\frac{1}{s}-\frac{1}{s+1}\right)+\frac{4}{s+1}$$
$\rightarrow$
$$£[y]=\frac{2e^{-3s}}{s}-\frac{2e^{-3s}}{s+1}+\frac{4}{s+1}$$
$\rightarrow$
$$\mathbf{y} = \mathbf{2}£^{-1}\left(\frac{e^{-3s}}{s}\right)-\mathbf{2}£^{-1}\left(\frac{e^{-3s}}{s+1}\right)+\mathbf{4}£^{-1}\left(\frac{1}{s+1}\right)$$

recalling that

$\boxed{£^{-1}[e^{-sa}F(s)] = u_a(t)f(t-a) \text{ , where } £^{-1}[F(s)] = f(t)}$
$£^{-1}[\frac{1}{s}] = 1$ **and** $£^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$
$\mathbf{y} = \mathbf{2u}_3(t)-\mathbf{2u}_3(t)\mathbf{e}^{-(t-3)}+\mathbf{4e}^{-t}$

It is important to note that the solution is valid for the values $t \neq 3$ **because** $\frac{dy}{dt}$ **does not exist at** $t = 3$

#14 on the page 581

To solve the initial value problem

$\frac{dy}{dt} = -2y + u_2(t)e^{-t}\ ,\ y(0) = 3$
may rewrite the equation as

$$\frac{dy}{dt}+\mathbf{2y} = \mathbf{u}_2(t)\mathbf{e}^{-t}$$

apply the Laplace Transform:

$£\left[\frac{dy}{dt}\right] + \mathbf{2}£[y] = £[u_2(t)e^{-t}]$

$-y(0) + s£[y] + 2£[y] = £[u_2(t)e^{-t}]$ .... **(1)**

Instead of calculating $£[u_2(t)e^{-t}]$ **directly, let us note that we know**

$£[u_a(t)f(t-a)] = e^{-sa}F(s)$, **where** $£[f] = F(s)$

Since, $£[e^{-t}] = \frac{1}{s+1}$

Therefore, we know that $£[u_2(t)e^{-(t-2)}] = e^{-2s}\left(\frac{1}{s+1}\right)$

We may write, $u_2(t)e^{-t} = u_2(t)e^{-t}e^2e^{-2} = u_2(t)e^{-(t-2)}e^{-2} = e^{-2}u_2(t)e^{-(t-2)}$

$£[u_2(t)e^{-t}] = £[e^{-2}u_2(t)e^{-(t-2)}] = \mathbf{e}^{-2}£[u_2(t)e^{-(t-2)}] = \mathbf{e}^{-2}\mathbf{e}^{-2s}\left(\frac{1}{s+1}\right)$

The equation (1) becomes

$-\mathbf{y}(0) + \mathbf{s}£[y] + \mathbf{2}£[y] = \mathbf{e}^{-2}\mathbf{e}^{-2s}\left(\frac{1}{s+1}\right)$

$-3 + (s+2)£[y] = e^{-2}e^{-2s}\left(\frac{1}{s+1}\right)$ **given that** $y(0) = 3$

$(s+2)£[y] = e^{-2}e^{-2s}\left(\frac{1}{s+1}\right) + 3$

$£[y] = \mathbf{e}^{-2}\left[\frac{e^{-2s}}{(s+1)(s+2)}\right] + \frac{3}{s+2}$

Use the method of Partial Fractions to obtain

$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$

$£[y] = \mathbf{e}^{-2}\left[\frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s+2}\right] + \frac{3}{s+2}$

$\rightarrow$

$\mathbf{y} = \mathbf{e}^{-2}£^{-1}\left[\frac{e^{-2s}}{s+1}\right] - \mathbf{e}^{-2}£^{-1}\left[\frac{e^{-2s}}{s+2}\right] + \mathbf{3}£^{-1}\left[\frac{1}{s+2}\right]$

or

$y = e^{-2}u_2(t)e^{-(t-2)} - e^{-2}u_2(t)e^{-2(t-2)} + 3e^{-2t}, \qquad t \neq 2$

Laplace Transforms of sine and cosine functions

Recall from the Integration by Parts that

$$\int \mathbf{e}^{at}\sin \boldsymbol{\omega}\mathbf{tdt} = -\frac{1}{a^2+\omega^2}(\omega(\cos\omega t)e^{at} - a(\sin\omega t)e^{at})$$

Therefore,

$$\int e^{-st}\sin\omega t dt = -\frac{1}{s^2+\omega^2}(\omega(\cos\omega t)e^{-st} + s(\sin\omega t)e^{-st})$$

$$\int_0^\infty \mathbf{e}^{-st}\sin \boldsymbol{\omega}\mathbf{tdt}$$

$$= \mathbf{Lim}_{b\to\infty}\left[-\frac{1}{s^2+\omega^2}(\omega(\cos\omega t)e^{-st} + s(\sin\omega t)e^{-st})\right]_0^b$$

$$= Lim_{b\to\infty}\left(-\frac{1}{s^2+\omega^2}(\omega(\cos\omega b)e^{-sb} + s(\sin\omega b)e^{-sb})\right) - \left(-\frac{1}{s^2+\omega^2}(\omega(\cos 0)e^0 + s(\sin 0)e^0)\right)$$

$e^{-sb} \to 0$ **as** $b \to \infty$

$\mathbf{-1} \le \cos \boldsymbol{\omega}\mathbf{b} \le \mathbf{1}$

therefore by squeeze theorem

$e^{-sb}\cos\omega b \to 0$ **as** $b \to \infty$

also $e^{-sb}\sin\omega b \to 0$ **as** $b \to \infty$

$$\int_0^\infty e^{-st}\sin\omega t dt = (0+0) - \left(-\frac{1}{s^2+\omega^2}(\omega+0)\right)$$

or

$$\int_0^\infty \mathbf{e}^{-st}\sin \boldsymbol{\omega}\mathbf{tdt} = \frac{\omega}{s^2+\omega^2}$$

That is $\boxed{£[\sin\omega t] = \frac{\omega}{s^2+\omega^2}}$

similarly $\boxed{\text{£}[\cos \omega t] = \frac{s}{s^2+\omega^2}}$

Let us derive and keep one more standard result that will come in quite handy later

If $\text{£}[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$

$$\text{£}[e^{at}f(t)]$$
$$=\int_0^\infty e^{at}f(t)e^{-st}dt$$
$$=\int_0^\infty e^{at-st}f(t)dt$$
$$=\int_0^\infty e^{-(s-a)t}f(t)dt$$

set $s - a = u$

$$\text{£}[e^{at}f(t)]=\int_0^\infty e^{-(u)t}f(t)dt=F(u) = F(s-a)$$

$$\boxed{\text{£}[e^{at}f(t)] = F(s-a)}$$

#32 on the page 595

To solve the initial value problem

$$\frac{d^2y}{dt^2} + 3y = u_4(t)\cos(5(t-4)) \ \ , \ y(0) = 0, \ y'(0) = -2$$

Where $\frac{dy}{dt} = y'$

$$£\left[\frac{d^2y}{dt^2}\right]+\mathbf{3}£[y]= £[u_4(t)\cos(5(t-4))]$$

$£\left[\frac{dy'}{dt}\right] + 3£[y] = £[u_4(t)\cos(5(t-4))]$ .... **(1)**

Recall from the above that

$$\boxed{£\left[\frac{dy}{dt}\right] = -y(0) + s£[y]}$$

therefore

$$£\left[\frac{dy'}{dt}\right] = -y'(0) + s£[y'] = -y'(0) + s(-y(0) + s£[y]) = -y'(0) - sy(0) + s^2£[y]$$

$£[u_a(t)f(t-a)] = e^{-sa}F(s)$, where $£[f] = F(s)$

therefore,

$$£[u_4(t)\cos(5(t-4))]= \mathbf{e}^{-4s}\frac{s}{s^2+25}=\frac{se^{-4s}}{s^2+25}$$

The equation (**1**) **becomes**

$$£\left[\frac{dy'}{dt}\right]+\mathbf{3}£[y]= £[u_4(t)\cos(5(t-4))]$$

$$-\mathbf{y}'(0)-\mathbf{sy}(0)+\mathbf{s}^2£[y]+\mathbf{3}£[y]=\frac{se^{-4s}}{s^2+25}$$

use $y(0) = 0$ **and** $y'(0) = -2$

$$-(-2)-\mathbf{0} + \mathbf{s}^2£[y]+\mathbf{3}£[y]=\frac{se^{-4s}}{s^2+25}$$

$\rightarrow$

$$\mathbf{2} + \mathbf{s}^2£[y]+\mathbf{3}£[y]=\frac{se^{-4s}}{s^2+25}$$

$\rightarrow$

$$\mathbf{s}^2\pounds[y]+\mathbf{3}\pounds[y]=\frac{se^{-4s}}{s^2+25}-\mathbf{2}$$

$\rightarrow$

$$(s^2+3)\pounds[y]=\frac{se^{-4s}}{s^2+25}-\mathbf{2}$$

$\rightarrow$

$$\pounds[y]=\frac{se^{-4s}}{(s^2+25)(s^2+3)}-\frac{2}{(s^2+3)}$$

$$\mathbf{y}=\pounds^{-1}\left[\frac{se^{-4s}}{(s^2+25)(s^2+3)}\right]-\pounds^{-1}\left[\frac{2}{(s^2+3)}\right]$$

Use the methods of Partial Fractions to obtain

$$\frac{s}{(s^2+25)(s^2+3)}=\frac{1}{22}\frac{s}{s^2+3}-\frac{1}{22}\frac{s}{s^2+25}$$

$$\mathbf{y}=\frac{1}{22}\pounds^{-1}\left[\frac{se^{-4s}}{s^2+3}\right]-\frac{1}{22}\pounds^{-1}\left[\frac{se^{-4s}}{s^2+25}\right]-\pounds^{-1}\left[\frac{2}{(s^2+3)}\right]$$

$$\pounds^{-1}\left[\frac{s}{s^2+3}\right]=\cos\sqrt{3}\,t \quad \rightarrow \quad \pounds^{-1}\left[\frac{se^{-4s}}{s^2+3}\right]=u_4(t)\cos\sqrt{3}\,(t-4)$$

$$\pounds^{-1}\left[\frac{s}{s^2+25}\right]=\cos\mathbf{5t} \quad \rightarrow \quad \pounds^{-1}\left[\frac{se^{-4s}}{s^2+25}\right]=\mathbf{u}_4(t)\cos\mathbf{5}(t-4)$$

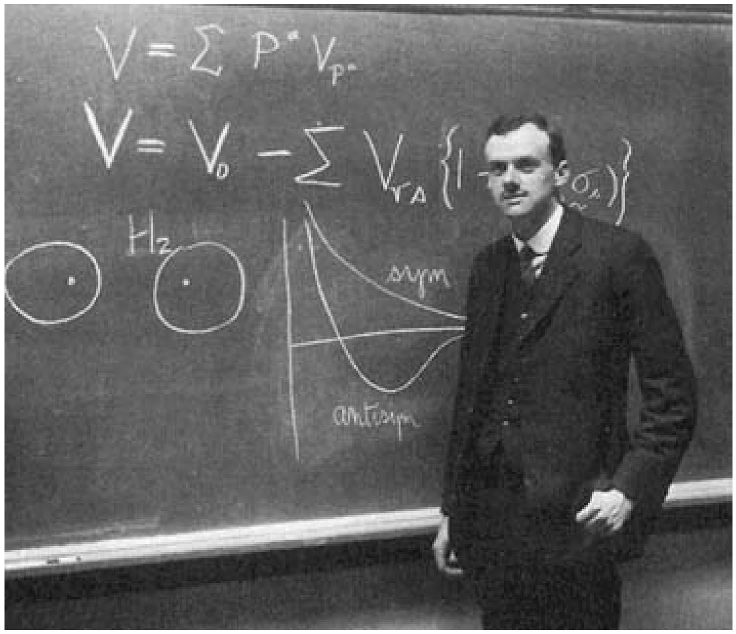
$$\pounds^{-1}\left[\frac{2}{(s^2+3)}\right]=\frac{2}{\sqrt{3}}\pounds^{-1}\left[\frac{\sqrt{3}}{(s^2+3)}\right]=\frac{2}{\sqrt{3}}\sin\sqrt{3}\,\mathbf{t}$$

$$y=\frac{1}{22}u_4(t)\cos\sqrt{3}\,(t-4)-\frac{1}{22}u_4(t)\cos 5(t-4)-\frac{2}{\sqrt{3}}\sin\sqrt{3}\,t\ ,\quad t\neq 4$$

Example of a very interesting and useful thing,

which is called the Dirac Delta Function but interesting thing is that it is NOT even a function.

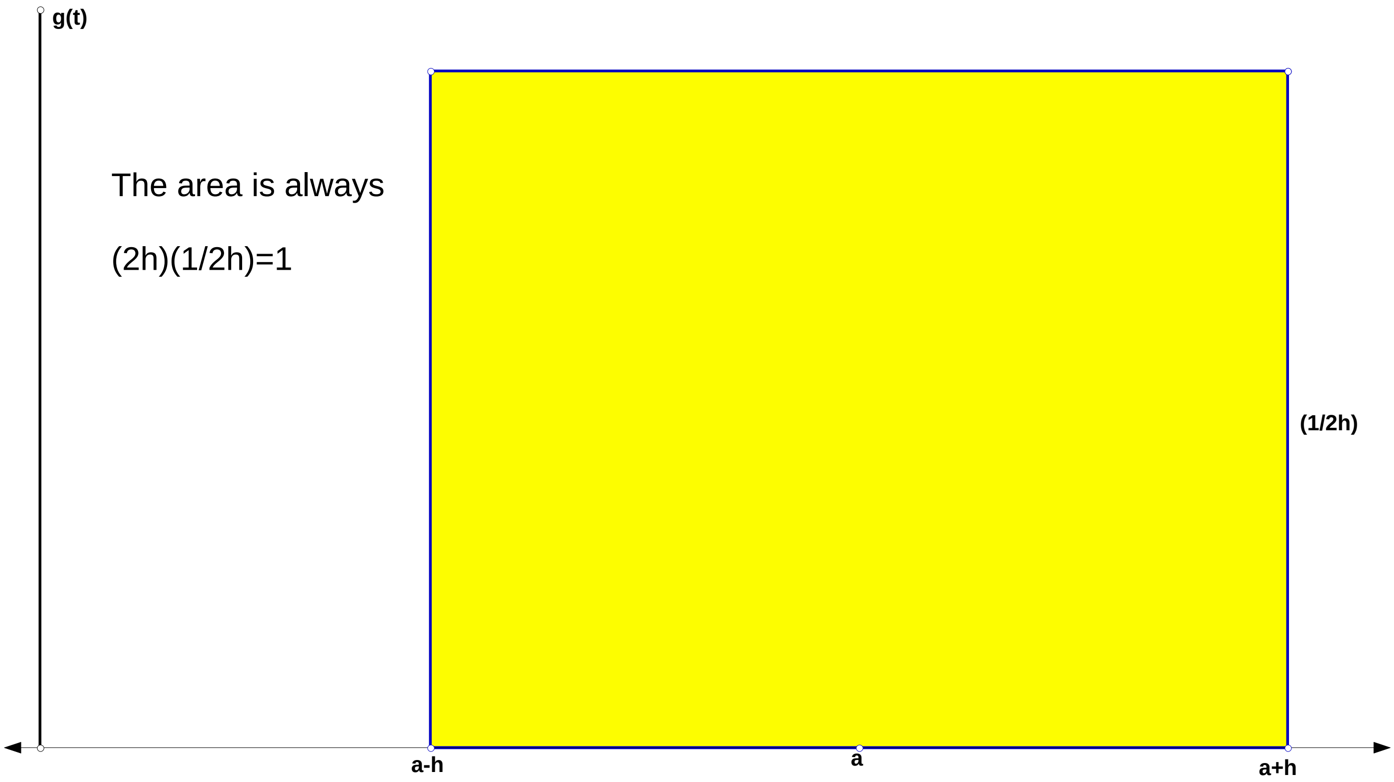
This method was developed by the Nobel Prize winning Physicist Paul Dirac



(**Photograph taken by another great mathematician Paul Halmos**)
in 1930s for dealing with impulsive functions.

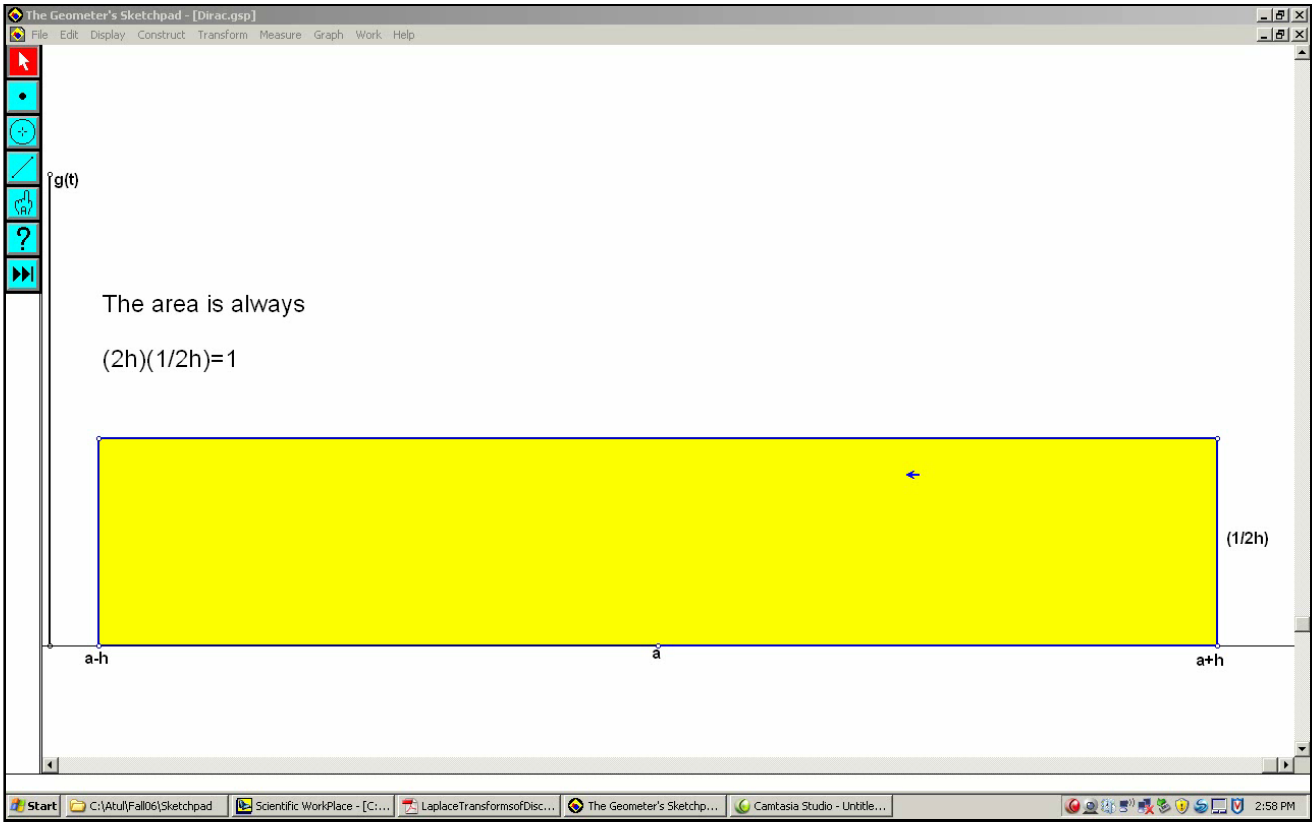
Let us look at the following function

$$\mathbf{g}_h(t) = \begin{cases} \frac{1}{2h} & \text{if} \quad a - h \le t \le a + h \\ 0 & \text{if} \quad t \in (-\infty, a-h) \cup (a+h, \infty) \end{cases}$$



Therefore, regardless of the value of h, the area of the rectangle shown above is 1

To see the area change as h decreases, double click the hand tool in the box below



or in other words

$$\int_{-\infty}^{\infty} \mathbf{g}_h(t)\mathbf{dt} = \mathbf{1}$$

whenever $h \neq 0$

For a real number a, **the Dirac Delta Function** $\delta_a(t) = Lim_{h\to 0} g_h(t)$

where

$$\mathbf{g}_h(t) = \begin{cases} \frac{1}{2h} & \text{if} \quad a-h \leq t \leq a+h \\ 0 & \text{if} \quad t \in (-\infty, a-h) \cup (a+h, \infty) \end{cases}$$

It looks awakward to say this but $\delta_a(t)$ **is 0 everywhere**, **except at** a **where it is** ∞

But we can find $£[\delta_a(t)]$

First of all let us find $£[g_h(t)]$

$$£[g_h(t)] = \int_0^{\infty} g_h(t)e^{-st}dt$$

$$= \int_{a-h}^{a+h} \frac{1}{2h} e^{-st}dt$$

$$=\frac{1}{2h}\int_{a-h}^{a+h} e^{-st}dt$$
$$=\frac{1}{2h}\left(-\frac{e^{-st}}{s}\Big|_{a-h}^{a+h}\right)$$
$$=\frac{1}{2h}\left(-\frac{e^{-s(a+h)}}{s}-\left(-\frac{e^{-s(a-h)}}{s}\right)\right)$$
$$=\frac{1}{2h}\left(-\frac{e^{-s(a+h)}}{s}+\frac{e^{-s(a-h)}}{s}\right)$$
$$=\frac{1}{2h}\left(-\frac{e^{-s(a+h)}}{s}+\frac{e^{-s(a-h)}}{s}\right)$$
$$=\frac{1}{2h}\left(\frac{-e^{-sa}e^{-sh}+e^{-sa}e^{sh}}{s}\right)$$
$$=\frac{e^{-sa}}{2s}\left(\frac{-e^{-sh}+e^{sh}}{h}\right)$$

$£[\delta_a(t)]$

$=Lim_{h\to 0}£[g_h(t)]$

$=Lim_{h\to 0}\frac{e^{-sa}}{2s}\left(\frac{-e^{-sh}+e^{sh}}{h}\right)$

$=\frac{e^{-sa}}{2s}Lim_{h\to 0}\left(\frac{-e^{-sh}+e^{sh}}{h}\right)$ $\frac{0}{0}$ **form, we may use the L'Hospital**

$=\frac{e^{-sa}}{2s}Lim_{h\to 0}\left(\frac{se^{-sh}+se^{sh}}{1}\right)$ **(differentiated the numerator &denominator wrt h)**

$=\frac{e^{-sa}}{2s}(2s)$

$=e^{-sa}$

Therefore $\boxed{£[\delta_a(t)] = e^{-sa}}$

Example:

#4 on the page 602

To solve the initial value problem

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+2y=-2\delta_2(t)\ ,\quad y(0)=2,\ \ y'(0)=0$$

Take the Laplace Transform of this differential equation

$$£\left[\frac{d^2y}{dt^2}\right]+\mathbf{2}£\left[\frac{dy}{dt}\right]+\mathbf{2}£[y]=-\mathbf{2}£[\delta_2(t)]$$

We had seen that

$$£\left[\frac{dy}{dt}\right] = -y(0) + s£[y]$$

$$£\left[\frac{d^2y}{dt^2}\right] = £\left[\frac{dy'}{dt}\right] = -\mathbf{y}'(0) + \mathbf{s}£[y'] = -\mathbf{y}'(0) + \mathbf{s}(-y(0) + s£[y]) = -\mathbf{y}'(0) - \mathbf{sy}(0) + \mathbf{s}^2£[y]$$

$$£[\delta_2(t)] = e^{-2s}$$

$$£\left[\frac{d^2y}{dt^2}\right] + \mathbf{2}£\left[\frac{dy}{dt}\right] + \mathbf{2}£[y] = -\mathbf{2}£[\delta_2(t)]$$

$\rightarrow$

$$-\mathbf{y}'(0) - \mathbf{sy}(0) + \mathbf{s}^2£[y] + \mathbf{2}(-y(0) + s£[y]) + \mathbf{2}£[y] = -\mathbf{2e}^{-2s}$$

$\rightarrow$

$$-\mathbf{0} - \mathbf{s}(2) + \mathbf{s}^2£[y] + \mathbf{2}(-(2) + s£[y]) + \mathbf{2}£[y] = -\mathbf{2e}^{-2s}$$

$\rightarrow$

$$-\mathbf{2s} + \mathbf{s}^2£[y] + \mathbf{2}(-(2) + s£[y]) + \mathbf{2}£[y] = -\mathbf{2e}^{-2s}$$

$\rightarrow$

$$-\mathbf{2s} + \mathbf{s}^2£[y] - \mathbf{4} + \mathbf{2s}£[y] + \mathbf{2}£[y] = -\mathbf{2e}^{-2s}$$

$\rightarrow$

$$(s^2 + 2s + 2)£[y] = \mathbf{2s} + \mathbf{4} - \mathbf{2e}^{-2s}$$

$\rightarrow$

$$£[y] = \frac{2s+4}{(s^2+2s+2)} - \mathbf{2}\frac{e^{-2s}}{(s^2+2s+2)}$$

or

$$\mathbf{y} = £^{-1}\left[\frac{2s+4}{(s^2+2s+2)}\right] - \mathbf{2}£^{-1}\left[\frac{e^{-2s}}{(s^2+2s+2)}\right]$$

Now we know that

$$£^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos\boldsymbol{\omega}\mathbf{t}$$

$$£^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \sin\boldsymbol{\omega}\mathbf{t}$$

therefore by using the techniques of completing the square, we may use

$$\boxed{£[e^{at}f(t)] = F(s-a)}$$

which means that

$£^{-1}[F(s-a)] = e^{at}f(t)$ **where** $£[f(t)] = F(s)$

$$\frac{2s+4}{(s^2+2s+2)} = \frac{2(s+2)}{s^2+2s+1+1} = \frac{2(s+2)}{(s+1)^2+1} = \frac{2(s+1)+2}{(s+1)^2+1} = \mathbf{2}\frac{(s+1)}{(s+1)^2+1} + \mathbf{2}\frac{1}{(s+1)^2+1}$$

$$£^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos\boldsymbol{\omega}\mathbf{t} \rightarrow £^{-1}\left[\frac{s+1}{(s+1)^2+1^2}\right] = \mathbf{e}^t\cos\mathbf{t}$$

$$£^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \sin\boldsymbol{\omega}\mathbf{t} \rightarrow £^{-1}\left[\frac{1}{(s+1)^2+1^2}\right] = \mathbf{e}^t\sin\mathbf{t}$$

Also, **from**

$$\boxed{£^{-1}[e^{-sa}F(s)] = u_a(t)f(t-a) \text{ , where } £^{-1}[F(s)] = f(t)}$$

and

$$£^{-1}\left[\frac{1}{(s+1)^2+1^2}\right] = e^{-t}\sin t$$

we have

$$£^{-1}\left[\frac{e^{-2s}}{(s^2+2s+2)}\right] = £^{-1}\left[e^{-2s}\frac{1}{(s+1)^2+1^2}\right] = u_2(t)e^{-(t-2)}\sin(t-2)$$

combining all of the above, **we have**

$$\mathbf{y} = £^{-1}\left[\frac{2s+4}{(s^2+2s+2)}\right] - \mathbf{2}£^{-1}\left[\frac{e^{-2s}}{(s^2+2s+2)}\right]$$

$\rightarrow$

$$\mathbf{y} = \mathbf{2}£^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + \mathbf{2}£^{-1}\left[\frac{1}{(s+1)^2+1}\right] - \mathbf{2}£^{-1}\left[e^{-2s}\frac{1}{(s+1)^2+1^2}\right]$$

$\rightarrow$

$$y = 2e^{-t}\cos t + 2e^{-t}\sin t - 2u_2(t)e^{-(t-2)}\sin(t-2) \text{ , } \quad t \neq 2$$

Let us study another operator called convolution which is often used for solving differential equations.

For functions f and g the convolution of f and g denoted by $f * g$ is defined by

$$f * g(t) = \int_0^t f(t-u)g(u)du$$

Example:

#2 on the page 610

To compute

$$e^{-at} * e^{-bt}$$

Use the definition to get

$$e^{-at} * e^{-bu}$$
$$= \int_0^t e^{-a(t-u)} e^{-bu} du$$
$$= \int_0^t e^{-a(t-u)-bu} du$$
$$= \int_0^t e^{-at+au-bu} du$$
$$= \int_0^t e^{-at+(a-b)u} du$$
$$= \frac{e^{-at+(a-b)u}}{a-b} \Big|_0^t$$

$$= \frac{e^{-at+(a-b)t}}{a-b} - \frac{e^{-at+(a-b)0}}{a-b}$$
$$= \frac{e^{-bt}}{a-b} - \frac{e^{-at}}{a-b}$$

The discussion in the book on the pages 604-605 illustrates the proof of the following

$\boxed{\pounds(f * g) = \pounds[f]\pounds[g] = F(s)G(s) \quad \text{or} \quad \pounds^{-1}(F(s)G(s)) = f * g}$ a result that is known as the Convolution Theorem

Applications

Example 1:

Consider the initial value problem

$$\frac{dy}{dt} + ay = f(t) \qquad\qquad y(0) = y_0$$

Take the Laplace Transform

$$\pounds\left[\frac{dy}{dt}\right] + a\pounds[y] = \pounds[f(t)]$$

$s\pounds[y] - y(0) + a\pounds[y] = F(s)$ where $\pounds[f(t)] = F(s)$

$$s\pounds[y] - y_0 + a\pounds[y] = F(s)$$
$$(s+a)\pounds[y] = y_0 + F(s)$$
$$\pounds[y] = \frac{y_0}{(s+a)} + \frac{F(s)}{(s+a)}$$

Remeber that $\pounds\left[e^{-at}\right] = \frac{1}{s+a}$

therefore

$$£\left[e^{-at} * f(t)\right] = £\left[e^{-at}\right] £[f(t)]$$

$$£\left[e^{-at} * f(t)\right] = \frac{1}{s+a}F(s) = \frac{F(s)}{s+a}$$

$$£^{-1}\left[\frac{F(s)}{s+a}\right] = e^{-at} * f(t)$$

For the solution

$$£[y] = \frac{y_0}{(s+a)} + \frac{F(s)}{(s+a)}$$

$$y = £^{-1}\left[\frac{y_0}{(s+a)}\right] + £^{-1}\left[\frac{F(s)}{(s+a)}\right]$$

$$y = y_0 £^{-1}\left[\frac{1}{(s+a)}\right] + £^{-1}\left[\frac{F(s)}{(s+a)}\right]$$

$$y = y_0 e^{-at} + e^{-at} * f(t)$$

is the solution of the initial value problem $\frac{dy}{dt} + ay = f(t) \qquad y(0) = y_0$

Example 2:

Convolutions may be used to solve integrodifferential equations as shown below.

The current in a deries curcuit may be modeled by Integrodifferential equations.

Let us take a simple example of an integrodifferential equation

To solve

$$\frac{dy}{dt} = 1 - \sin t - \int_0^t y(u)du \quad , \quad y(0) = 0$$

First note that

$$\int_0^t y(u)du = 1 * y(t)$$

therefore

$$£\left[\int_0^t y(u)du\right] = £[1 * y(t)] = £[1]£[y] = \frac{1}{s}£[y]$$

Taking the Laplace Transform of the differential equation

$$\frac{dy}{dt} = 1 - \sin t - \int_0^t y(u)du \quad , \quad y(0) = 0$$

$$£\left[\frac{dy}{dt}\right] = £[1] - £[\sin t] - £\left[\int_0^t y(u)du\right]$$

$$s£[y] - y(0) = \frac{1}{s} - \frac{1}{s^2+1} - \frac{1}{s}£[y]$$, table 6.1 (page 620 and $£\left[\int_0^t y(u)du\right] = \frac{1}{s}£[y]$

$$s£[y] = \frac{1}{s} - \frac{1}{s^2+1} - \frac{1}{s}£[y]$$

$$s£[y] + \frac{1}{s}£[y] = \frac{1}{s} - \frac{1}{s^2+1}$$

$$\left(\frac{s^2+1}{s}\right)£[y] = \frac{s^2+1-s}{s(s^2+1)}$$

$$£[y] = \left(\frac{s}{s^2+1}\right)\frac{s^2+1-s}{s(s^2+1)}$$

$$£[y] = \frac{s^2+1-s}{(s^2+1)^2}$$

$$£[y] = \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}$$

$$y = £^{-1}\left[\frac{1}{s^2+1}\right] - £^{-1}\left[\frac{s}{(s^2+1)^2}\right]$$

$y = \sin t - t\sin t$ TABLE 6.1, this table will be included in the final exam