In chapter 3, we worked on the second order homogeneous equations,

let us proceed to chapter 4 and solve the second order non homogeneous equations of the type

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

Our methods will rely a theoretical result that if

y_1 and y_2 are two linearly independent solutions of the homogeneous equation $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$

and $\Psi(t)$ is a particular solution of $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$,

then every solution y of $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$ has to be of the form $y = c_1y_1 + c_2y_2 + \Psi(t)$ for some choice of constants c_1 and c_2

For now, we shall limit ourselves to the cases where $p(t)$ and $q(t)$ are constants and

$g(t)$ is either

a) a polynomial

b) is of the form e^{ax}

c) is of the form $\sin ax$ or $\cos ax$

d) is of the form $x^{\alpha}e^{ax}\cos bx$, $x^{\alpha}e^{ax}\sin bx$

The method that we are going to use is called the method of undetermined coefficients.

The homogeneous differential equations are related with harmonic oscillators (look at the page 193 of the text book)

The non homogeneous systems that we are using in this chapter cover some cases when we are ading an external force that will affect the motion of a harmonic oscillator. The function $g(t)$ on the right hand side of the differential equation shows the inclusion of the external force. (look at the pages 383-383 of the text book)

Example 1

To solve the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = e^{5t}$$

First let us find linearly independent solutions of the homogeneous equation $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$

as we did before, for a solution e^{st} , we must have
$s^2 - 2s - 3 = 0$
that is $(s-3)(s+1) = 0$

or $s = -1$ and $s = 3$

therefore e^{-t} and e^{3t} are two linearly independent solutions of the homogeneous equation $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$

For a particular solution of $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = e^{5t}$
let us try $\Psi(t) = ke^{5t}$, what the book calls a lucky guess,

where k is a constant

to determine the value of k, we can substite $y = ke^{5t}$
in $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = e^{5t}$ and solve for k
with $y = ke^{5t}$

$$\frac{dy}{dt} = 5ke^{5t}$$
$$\frac{d^2y}{dt^2} = 25ke^{5t}$$

substitute these values in

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = e^{5t}$$

$$25ke^{5t} - 2(5ke^{5t}) - 3(ke^{5t}) = e^{5t}$$

$$25ke^{5t} - 10ke^{5t} - 3ke^{5t} = e^{5t}$$

$$25ke^{5t} - 10ke^{5t} - 3ke^{5t} - e^{5t} = 0$$

$$12ke^{5t} - e^{5t} = 0$$

$$(12k - 1)e^{5t} = 0$$

$$12k - 1 = 0$$

$$k = \frac{1}{12}$$

Therefore

A notation for a particular solution is y_p

in this example, we have

$$y_p = \frac{1}{12}e^{5k}$$

General solution of the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = e^{5t}$$

is

$$\boxed{y = c_1e^{-t} + c_2e^{3t} + \frac{1}{12}e^{5t}}$$

Let us check the solution

$$\frac{dy}{dt} = \frac{d}{dt}\left(c_1e^{-t} + c_2e^{3t} + \frac{1}{12}e^{5t}\right) = \frac{1}{12}(-12c_1e^{-6t} + 36c_2e^{-2t} + 5)e^{5t}$$
$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{12}(-12c_1e^{-6t} + 36c_2e^{-2t} + 5)e^{5t}\right) = c_1e^{-t} + 9c_2e^{3t} + \frac{25}{12}e^{5t}$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y$$
$$=$$

$$c_1e^{-t} + 9c_2e^{3t} + \frac{25}{12}e^{5t} - \frac{2}{12}(-12c_1e^{-6t} + 36c_2e^{-2t} + 5)e^{5t} - 3\left(c_1e^{-t} + c_2e^{3t} + \frac{1}{12}e^{5t}\right) =$$
$$e^{5t}$$

Example 2:

#18 on the page 394

For the differential equation:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{-4t}$$

a) compute the general solution

b) compute the solutions with $y(0) = y'(0) = 0$

and

c) to describe the long term behavior of solutions in a brief paragraph

First,

let us solve the homogeneous equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 0$$

solve

$$s^2 + 4s + 20 = 0$$

Solutions are: $\{s = -2 + 4i\}, \{s = -2 - 4i\}$ by the Quadratic Formula

linearly independent solutions of the above homogeneous system are

$e^{-2t}\cos 4t$ and $e^{-2t}\sin 4t$

now assume that

$y_p = ke^{-4t}$ is a particular solution of

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{-4t}$$ .... (1)

$$\frac{dy_p}{dt} = -4ke^{-4t}$$
$$\frac{d^2y_p}{dt^2} = 16ke^{-4t}$$

to find the value of k, substituite these values in the differential equation (1)

$$16ke^{-4t} + 4(-4ke^{-4t}) + 20ke^{-4t} = e^{-4t}$$
$$20ke^{-4t} = e^{-4t}$$
$$20k = 1$$
$$k = \frac{1}{20}$$

General solution of $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{-4t}$

is

$$y = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t) + \frac{1}{20}e^{-4t}$$

b)

To find the solution corresponding to

$$y(0) = y'(0) = 0$$

with

$$y(t) = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t) + \frac{1}{20}e^{-4t}$$
$$y(0) = e^{-2(0)}(c_1 \cos 4(0) + c_2 \sin 4(0)) + \frac{1}{20}e^{-4(0)} = c_1 + \frac{1}{20}$$

$$y'(t) = -e^{-2t}(c_1 \cos 4t + c_2 \sin 4t) + e^{-2t}(-4c_1 \sin 4t + 4c_2 \cos 4t) - \frac{1}{5}e^{-4t}$$
$$y'(0)$$
$$= -e^{-2(0)}(c_1 \cos 4(0) + c_2 \sin 4(0)) + e^{-2(0)}(-4c_1 \sin 4(0) + 4c_2 \cos 4(0)) - \frac{1}{5}e^{-4(0)}$$
$$= c_1 + 4c_2 - \frac{1}{5}$$

$$y(0) = c_1 + \frac{1}{20}$$
$$y(0) = 0 \to c_1 + \frac{1}{20} = 0 \to c_1 = -\frac{1}{20}$$

$$y'(0) = 0 \to c_1 + 4c_2 - \frac{1}{5} = 0 \to -\frac{1}{20} + 4c_2 - \frac{1}{5} = 0 \to 4c_2 - \frac{1}{4} = 0 \to c_2 = \frac{1}{16}$$

$y = e^{-2t}\left(-\frac{1}{20}\cos 4t + \frac{1}{16}\sin 4t\right) + \frac{1}{20}e^{-4t}$ is the solution corresponding to

the conditions $y(0) = y'(0) = 0$

c)

$$y = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t) + \frac{1}{20}e^{-4t}$$

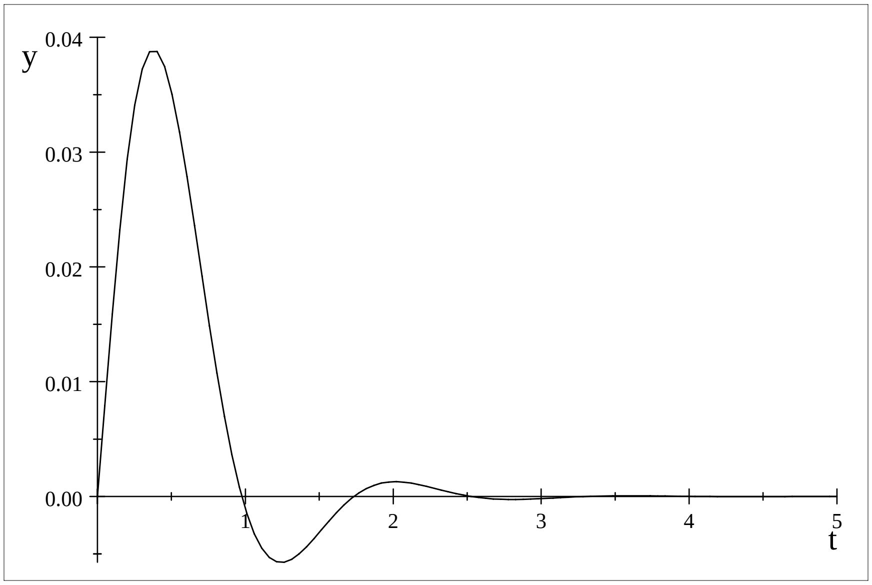
Since

$|\sin 4t| \leq 1$ and $|\cos 4t| \leq 1$
and $e^{-2t} \to 0$ as well as $e^{-4t} \to 0$ as $t \to \infty$

as $t \to \infty$, $y \to 0$

In particular

$$y = e^{-2t}\left(-\frac{1}{20}\cos 4t + \frac{1}{16}\sin 4t\right) + \frac{1}{20}e^{-4t}$$



In the chapter 4, we are taking up the solutions of the differential equations of the type

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

where p and q are constants and $g(t)$ is one of the specific functions that are listed in the table below.

The table summarises the book's approach of using the particular solution for these specific functions

$g(t)$	form of particular solution	condition
t^n	$a_0 + a_1 t + a_2 t^2 + \ldots.. + a_1 t^n$	0 is not a root of $s^2 + ps + q = 0$
t^n	$t^\alpha(a_0 + a_1 t + a_2 t^2 + \ldots.. + a_1 t^n)$	0 is a root of multiplicity α of $s^2 + ps + q = 0$
$e^{\beta t}$	$ce^{\beta t}$	β is not a root of $s^2 + ps + q = 0$
$e^{\beta t}$	$ct^\alpha e^{\beta t}$	β is a root of multiplicity α of $s^2 + ps + q = 0$
$t^n e^{\beta t}$	$c(a_0 + a_1 t + a_2 t^2 + \ldots.. + a_1 t^n)e^{\beta t}$	β is not a root of $s^2 + ps + q = 0$
$t^n e^{\beta t}$	$ct^\alpha(a_0 + a_1 t + a_2 t^2 + \ldots.. + a_1 t^n)e^{\beta t}$	β is a root of multiplicity α of $s^2 + ps + q = 0$
$\cos \beta t$	$c_1 \cos \beta t + c_2 \sin \beta t$	$i\beta$ is not a root of $s^2 + ps + q = 0$
$\cos \beta t$	$t^\alpha(c_1 \cos \beta t + c_2 \sin \beta t)$	$i\beta$ is a root of multiplicity α of $s^2 + ps + q = 0$

the cases involving $\sin \beta t$ are similar to the ones shown above

Example 3:

#24 on the page 395

For the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = -8$$

a) to find the general solution

b) to find the solution with $y(0) = y'(0) = 0$

Solution:

For $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = 0$

solve:

$s^2 + 4s + 6 = 0$ by using the quadratic formula to get

the Solutions $i\sqrt{2} - 2, -i\sqrt{2} - 2$

and note that

$e^{-2t}\cos\sqrt{2}\,t$ and $e^{-2t}\sin\sqrt{2}\,t$

are two linearly independent solutions of the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = 0$$

Based on the suggestions in the table above

let us try $y_p(t) = a$ as a particular solution, where a is a constant.

$$\frac{da}{dt} = 0$$
$$\frac{d^2a}{dt^2} = 0$$

substituting these values in $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = -8$

we get

$$6a = -8$$

that is

$a = -\frac{4}{3}$

therefore general solution of $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = -8$

is

$y = c_1 e^{-2t} \cos\sqrt{2}\,t + c_2 e^{-2t} \sin\sqrt{2}\,t - \frac{4}{3}$

b)

for the solutions with $y(0) = y'(0) = 0$

We have

$$y(t) = c_1 e^{-2t} \cos\sqrt{2}\,t + c_2 e^{-2t} \sin\sqrt{2}\,t - \frac{4}{3}$$
$$y(0) = c_1 - \frac{4}{3}$$

$$y'(t) = -2c_1 e^{-2t} \cos\sqrt{2}\,t - c_1\sqrt{2}\,e^{-2t} \sin\sqrt{2}\,t - 2c_2 e^{-2t} \sin\sqrt{2}\,t + c_2\sqrt{2}\,e^{-2t} \cos\sqrt{2}\,t$$
$$y'(0) = -2c_1 + \sqrt{2}\,c_2$$

$$y(0) = 0 \rightarrow c_1 - \frac{4}{3} = 0 \rightarrow c_1 = \frac{4}{3}$$

with the above values

$$y'(0) = 0 \rightarrow -2c_1 + \sqrt{2}\,c_2 = 0 \rightarrow -\frac{8}{3} + \sqrt{2}\,c_2 = 0 \rightarrow c_2 = -\frac{8}{3\sqrt{2}}$$

therefore solution with $y(0) = y'(0) = 0$

is

$$y(t) = \frac{4}{3}e^{-2t}\cos\sqrt{2}\,t - \frac{8}{3\sqrt{2}}e^{-2t}\sin\sqrt{2}\,t - \frac{4}{3}$$

Example 4:

#30 on the page 395

For the differential equation

$$\frac{d^2y}{dt^2} + 2y = e^{-t}$$

a) compute the general solution

b) compute the solution with $y(0) = y'(0) = 0$

c) give a rough sketch and describe in a brief paragraph the long term behavior of the solution in part b)

Solution:

Because the solutions of $s^2 + 2 = 0$ are $\pm\sqrt{2}\,i$
$\cos\sqrt{2}\,t$ and $\sin\sqrt{2}\,t$

are linearly independent solutions of $\frac{d^2y}{dt^2} + 2y = 0$

assume that $y_p = ke^t$ is a particular solution of $\frac{d^2y}{dt^2} + 2y = -e^t$

$$\frac{d^2y_p}{dt^2} = ke^t$$

substitute the above values in $\frac{d^2y}{dt^2} + 2y = -e^t$
$ke^t + 2ke^t = -e^t \rightarrow k = -\frac{1}{3}$

General solution of $\frac{d^2y}{dt^2} + 2y = -e^t$

is

$$y(t) = c_1 \cos\sqrt{2}\,t + c_2 \sin\sqrt{2}\,t - \frac{1}{3}e^t$$

b)
$$y'(t) = -\sqrt{2}\,c_1 \sin\sqrt{2}\,t + \sqrt{2}\,c_2 \cos\sqrt{2}\,t - \frac{1}{3}e^t$$

$$y(0) = c_1 - \frac{1}{3}$$

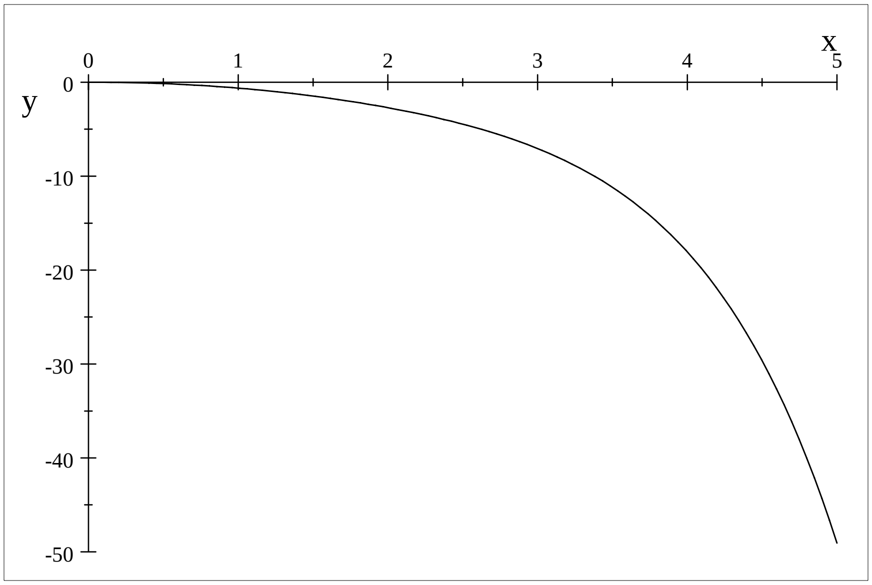
$$y'(0) = \sqrt{2}\,c_2 - \frac{1}{3}$$

$$y(0) = 0 \rightarrow c_1 = \frac{1}{3}$$
$$y'(0) = 0 \rightarrow \sqrt{2}\,c_2 - \frac{1}{3} = 0 \rightarrow c_2 = \frac{1}{3\sqrt{2}}$$

the solution with $y(0) = y'(0) = 0$

is

$$y = \frac{1}{3}\cos\sqrt{2}\,t + \frac{1}{3\sqrt{2}}\sin\sqrt{2}\,t - \frac{1}{3}e^t$$



as $t \to \infty$, the $y(t) \to -\infty$

Example 5:

#42 on the page 398

For the differential equation

$$\frac{d^2y}{dt^2} + 4y = 6 + t^2 + e^t$$

a) to compute the general solution

b) compute the solution with $y(0) = y'(0) = 0$

c) briefly the long term behavior of the solution in part b)

Solution:

Note that $\cos 2t$ and $\sin 2t$ are two linearly independent solutions of

$$\frac{d^2y}{dt^2} + 4y = 0$$

For

a general solution of

$$\frac{d^2y}{dt^2} + 4y = 6 + t^2 + e^t$$

let us proceed along the lines of the exercise #36

that is

For $6 + t^2$, form of the particular integral is $a_0 + a_1t + a_2t^2$

and

for e^t, form of the particular integral is ke^t

We may try $y_p = a_0 + a_1t + a_2t^2 + ke^t$ as a particular solution

$$\left(\frac{dy_p}{dt}\right) = a_1 + 2a_2t + ke^t$$
$$\left(\frac{d^2y_p}{dt^2}\right) = 2a_2 + ke^t$$

substitute these values in $\frac{d^2y}{dt^2} + 4y = 6 + t^2 + e^t$

to obtain

$2a_2 + ke^t + 4(a_0 + a_1 t + a_2 t^2 + ke^t) = 6 + t^2 + e^t$

$\rightarrow$

$2a_2 + ke^t + 4a_0 + 4a_1 t + 4a_2 t^2 + 4ke^t = 6 + t^2 + e^t$
$\rightarrow$
$(4a_0 + 2a_2) + 4a_1 t + 4a_2 t^2 + 5ke^t = 6 + t^2 + e^t$
$\rightarrow$
$4a_0 + 2a_2 = 6$
$4a_1 = 0$
$4a_2 = 1 \rightarrow a_2 = \frac{1}{4}$
$5k = 1 \rightarrow k = \frac{1}{5}$
$4a_0 + 2a_2 = 6 \rightarrow 4a_0 + 2\left(\frac{1}{4}\right) = 6 \rightarrow 4a_0 + \frac{1}{2} = 6 \rightarrow 4a_0 = \frac{11}{2} \rightarrow a_0 = \frac{11}{8}$

therefore general solution of

$\frac{d^2y}{dt^2} + 4y = 6 + t^2 + e^t$ is

$y = c_1 \cos 2t + c_2 \sin 2t + \frac{11}{8} + \frac{1}{4}t^2 + \frac{1}{5}e^t$

b)

To apply the conditions $y(0) = y'(0) = 0$
$y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{11}{8} + \frac{1}{4}t^2 + \frac{1}{5}e^t$

$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{1}{2}t + \frac{1}{5}e^t$

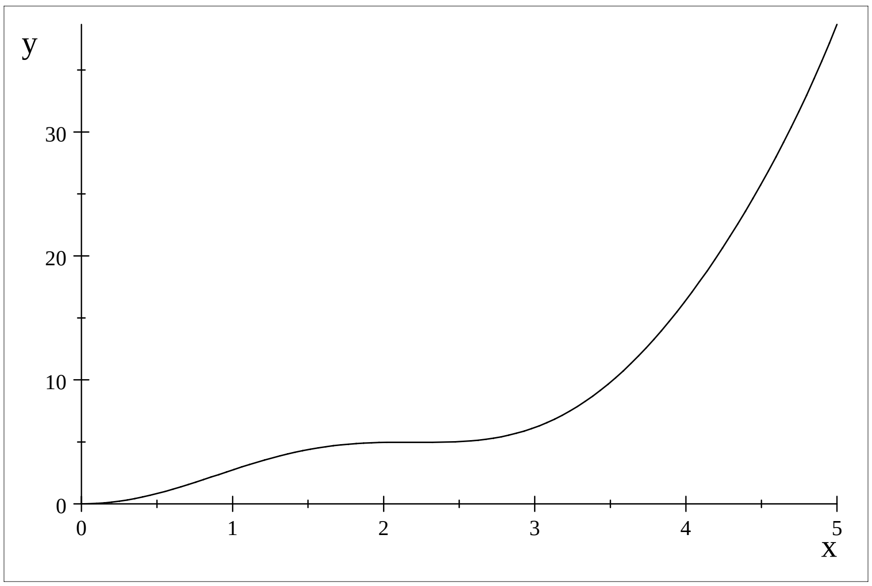
$y(0) = c_1 + \frac{11}{8} + \frac{1}{5}$
$y(0) = 0 \rightarrow c_1 + \frac{63}{40} = 0 \rightarrow c_1 = -\frac{63}{40}$
$y'(0) = 2c_2 + \frac{1}{5}$

$y'(0) = 0 \rightarrow 2c_2 + \frac{1}{5} = 0 \rightarrow c_2 = -\frac{1}{10}$

the solution with $y(0) = y'(0) = 0$

is
$y(t) = -\frac{63}{40}\cos 2t - \frac{1}{10}\sin 2t + \frac{11}{8} + \frac{1}{4}t^2 + \frac{1}{5}e^t$

$y(t) = -\frac{63}{40}\cos 2t - \frac{1}{10}\sin 2t + \frac{11}{8} + \frac{1}{4}t^2 + \frac{1}{5}e^t$



$\frac{1}{5}e^t$ will dominate the long term behavior and this solution goes to ∞ fast as $t \rightarrow \infty$

Example 6:

#14 on the page 407

To solve the initial value problem:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2\cos 2t$$

$y(0) = y'(0) = 0$

For

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$$

solve $s^2 + 2s + 1 = 0$

$(s+1)^2 = 0$

$s = -1$ is a repeated root

e^{-t} and te^{-t} are linearly independent solutions of $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$

For, $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2\cos 2t$ we may try
$y_p = c_1 \cos 2t + c_2 \sin 2t$

$$\frac{dy_p}{dt} = -c_1 2\sin 2t + c_2 2\cos 2t = -2c_1 \sin 2t + 2c_2 \cos 2t$$
$$\frac{d^2y_p}{dt^2} = -c_1 4\cos 2t - c_2 4\sin 2t = -4c_1 \cos 2t - 4c_2 \sin 2t$$

substitite in $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2\cos 2t$

$$-4c_1 \cos 2t - 4c_2 \sin 2t + 2(-2c_1 \sin 2t + 2c_2 \cos 2t) + c_1 \cos 2t + c_2 \sin 2t = 2\cos 2t$$

$$-4c_1 \cos 2t - 4c_2 \sin 2t - 4c_1 \sin 2t + 4c_2 \cos 2t + c_1 \cos 2t + c_2 \sin 2t = 2\cos 2t$$

$(-3c_1+4c_2)\cos 2t+(-4c_1-3c_2)\sin 2t = 2\cos 2t$

OR

$-3c_1+4c_2 = 2$ .... (1)
$-4c_1-3c_2 = 0$ .... (2)

multiply the equation number (1) by 3 and the equation number (2) by 4 and add
$c_1 = -\frac{6}{25}$

substitute $c_1 = -\frac{6}{25}$ in (2)

$-4\left(-\frac{6}{25}\right)-3c_2 = 0 \rightarrow \frac{24}{25}-3c_2 = 0 \rightarrow c_2 = \frac{8}{25}$

OR you may use the matrix method

$\begin{bmatrix} -3 & 4 & 2 \\ -4 & -3 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & -\frac{6}{25} \\ 0 & 1 & \frac{8}{25} \end{bmatrix}$

General solution is

$y = a_1e^{-t}+a_2te^{-t}-\frac{6}{25}\cos 2t+\frac{8}{25}\sin 2t$

with the conditions:

$y(0) = 0$ and $y'(0) = 0$

$y(t) = a_1e^{-t}+a_2te^{-t}-\frac{6}{25}\cos 2t+\frac{8}{25}\sin 2t$
$y'(t) = -a_1e^{-t}+a_2e^{-t}-a_2te^{-t}+\frac{12}{25}\sin 2t+\frac{16}{25}\cos 2t$

$$y(0) = 0 \to a_1 - \frac{6}{25} = 0 \to a_1 = \frac{6}{25}$$

$$y'(0) = 0 \to -a_1 + a_2 + \frac{16}{25} = 0 \to -\frac{6}{25} + a_2 + \frac{16}{25} = 0 \to a_2 = -\frac{10}{25} \to a_2 = -\frac{2}{5}$$

the solution of the initial value problem is

$$y(t) = \frac{6}{25} e^{-t} - \frac{2}{5} te^{-t} - \frac{6}{25} \cos 2t - \frac{8}{25} \sin 2t$$

Example 7:

To find the general solution of

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{-2t} \sin 4t$$

For

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 0$$

solve $s^2{+}4s+20=0$ by using the Quadratic Formula

$s^2+4s+20=0$, Solution is: $-2+4i,-2-4i$

linearly independent solutions are

$e^{-2t}\cos 4t$ and $e^{-2t}\sin 4t$

When we look at $g(t)=e^{-2t}\sin 4t$

first, it turns out to be the one of the solutions of the homogeneous part
second, it is not covered in any of the cases listed below

$g(t)$	form of particular solution	condition
t^n	$a_0+a_1t+a_2t^2+\ldots\ldots+a_1t^n$	0 is not a root of $s^2+ps+q=0$ (auxiliary eq)
t^n	$t^\alpha(a_0+a_1t+a_2t^2+\ldots\ldots+a_1t^n)$	0 is a root of multiplicity α of $s^2+ps+q=0$
$e^{\beta t}$	$ce^{\beta t}$	β is not a root of $s^2+ps+q=0$
$e^{\beta t}$	$ct^\alpha e^{\beta t}$	β is a root of multiplicity α of $s^2+ps+q=0$
$t^ne^{\beta t}$	$c(a_0+a_1t+a_2t^2+\ldots\ldots+a_1t^n)e^{\beta t}$	β is not a root of $s^2+ps+q=0$
$t^ne^{\beta t}$	$ct^\alpha(a_0+a_1t+a_2t^2+\ldots\ldots+a_1t^n)e^{\beta t}$	β is a root of multiplicity α of $s^2+ps+q=0$
$\cos\beta t$	$c_1\cos\beta t+c_2\sin\beta t$	$i\beta$ is not a root of $s^2+ps+q=0$
$\cos\beta t$	$t^\alpha(c_1\cos\beta t+c_2\sin\beta t)$	$i\beta$ is a root of multiplicity α of $s^2+ps+q=0$

Return to $e^{-2t}\sin 4t$

It is the imaginary part of the expression $e^{-2t}(\cos 4t+i\sin 4t)$
which is the same as the imaginary part of $e^{-2t}e^{4it}=e^{(-2+4i)t}$

since, $-2+4i$ is a solution of multiplicity 1 of the auxiliary equation, we may try $y_p = te^{2t}(c_1 \cos 4t + c_2 \sin 4t)$ as a particular solution of

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{-2t}\sin 4t$$

and proceed with routine substitution and find the values of c_1 and c_2

but a smarter strategy in this case will be to take up the imaginary part of

$y_p = ate^{(-2+4i)t}$ as a particular solution

Look at the complex version of the differential equation

$$\frac{d^2y_p}{dt^2} + 4\frac{dy_p}{dt} + 20y_p = e^{(-2+4i)t}$$

we are putting t as an extra factor because $-2+4i$ is a root of (multiplicity 1)

of the auxiliary equation.

$$y_p = ate^{(-2+4i)t}$$

$$\frac{dy_p}{dt} = ae^{(-2+4i)t} + at(-2+4i)e^{(-2+4i)t}$$

$$\frac{dy_p}{dt} = (a + at(-2+4i))e^{(-2+4i)t}$$

$$\frac{d^2y_p}{dt^2} = (a(-2+4i))e^{(-2+4i)t} + (a + at(-2+4i))(-2+4i)e^{(-2+4i)t}$$

$$\frac{d^2y_p}{dt^2} = ((a(-2+4i)) + (a + at(-2+4i))(-2+4i))e^{(-2+4i)t}$$

put these values in

$$((a(-2+4i)) + (a + at(-2+4i))(-2+4i))e^{(-2+4i)t} + 4(a + at(-2+4i))e^{(-2+4i)t} + 20ate^{(-2+4i)t}$$

$$[((a(-2+4i)) + (a - 2at + 4iat)(-2+4i)) + 4(a + at(-2+4i)) + 20at]e^{(-2+4i)t} = e^{(-2+4i)t}$$

$$[-2a + 4ia - 2a + 4ia + 4at - 8ati - 8ati - 16at + 4a - 8at + 16ati + 20at]e^{(-2+4i)t} = e^{(-2+4i)t}$$

after cancellations we get

$$[4ia + 4ia]e^{(-2+4i)t} = e^{(-2+4i)t}$$

that is

$$8ia = 1 \rightarrow a = \frac{1}{8i} = \frac{i}{8i^2} = -\frac{1}{8}i$$

therefore a particular solution is obtained by finding the imaginary part of
$y_p = ate^{(-2+4i)t}$

that is
$$y_p = -\frac{1}{8}ite^{-2t}(\cos 4t + i\sin 4t) = -\frac{1}{8}ite^{-2t}\cos 4t + \frac{1}{8}te^{-2t}\sin 4t$$

therefore $-\frac{1}{8}te^{-2t}\cos 4t$ the imaginary part of the above expression,
is a particular solution

General solution of

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt} + 20y = e^{-2t}\sin 4t$$

is

$$y = e^{-2t}(a_1\cos 4t + a_2\sin 4t) - \frac{1}{8}te^{-2t}\cos 4t$$

Please read the sections 4.3,4.4,4.5 in the text book

$$y_p = te^{2t}(c_1 \cos 4t + c_2 \sin 4t)$$
$$\frac{dy_p}{dt} = e^{2t}(c_1 \cos 4t + c_2 \sin 4t) + 2te^{2t}(c_1 \cos 4t + c_2 \sin 4t) + te^{2t}(-4c_1 \sin 4t + 4c_2 \cos 4t)$$

OR
$$\frac{dy_p}{dt} = e^{2t}(c_1 \cos 4t + c_2 \sin 4t) + te^{2t}(2(c_1 \cos 4t + c_2 \sin 4t) + (-4c_1 \sin 4t + 4c_2 \cos 4t))$$
OR
$$\frac{dy_p}{dt} = e^{2t}(c_1 \cos 4t + c_2 \sin 4t) + te^{2t}((2c_1 + 4c_2)\cos 4t + (-4c_1 + 2c_2)\sin 4t)$$
OR

$$\frac{dy_p}{dt} = (c_1 + 2(c_1 + 2c_2)t)e^{2t} \cos 4t + (c_2 + 2(-2c_1 + c_2)t)e^{2t} \sin 4t$$
$$\frac{dy_p}{dt} = ((c_1 + 2(c_1 + 2c_2)t)\cos 4t + (c_2 + 2(-2c_1 + c_2)t)\sin 4t)e^{2t}$$

therefore, we have

$$\frac{d^2y_p}{dt^2} = (2(c_1 + 2c_2)\cos 4t - 4(c_1 + 2(c_1 + 2c_2)t)\sin 4t + (2(-2c_1 + c_2))\sin 4t + 4)e^{2t}$$