The posted lessons are part of the Differential Equations course that I taught at Montgomery College in Germantown Maryland.

The lessons are written according to *Differential Equations* , Third Edition, by Blanchard, Devaney, and Hall , Brooks/Cole as the text book adopted for the class.

For any questions, comments or objections

You may write to me at

atulnarainroy@gmail.com

Lesson 3:

Please note that this section will require a very extensive reading from the T

Section 1.6:

In this section, we shall work on differential equations of the type

$\frac{dy}{dt} = f(y)$ **, also termed as autonomous differential equation.**

Recall that the equilibrium points of the above differential equation are the values of y

for which $f(y) = 0$

for example:

The equilibrium points of the differential equation $\frac{dy}{dt} = y^2 - y - 6$

Here $f(y) = y^2 - y - 6$

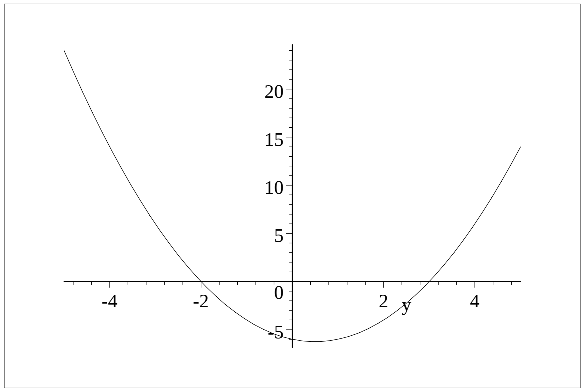
and $y^2 - y - 6 = 0$

if $(y-3)(y+2) = 0$

if $y = -2$ **or** $y = 3$

Equilibrium points are -2 **and** 3

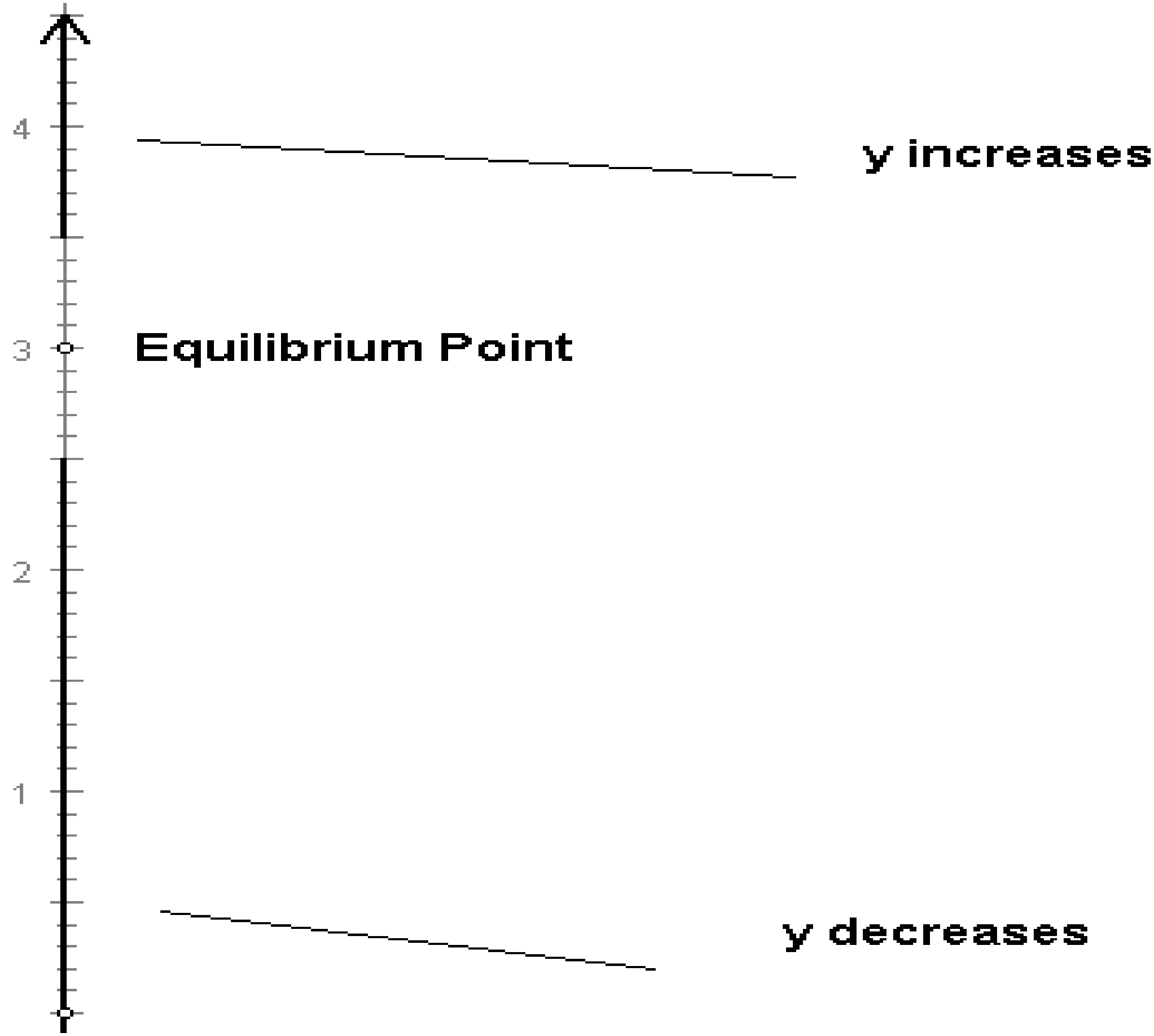
From a graph of $y^2 - y - 6$



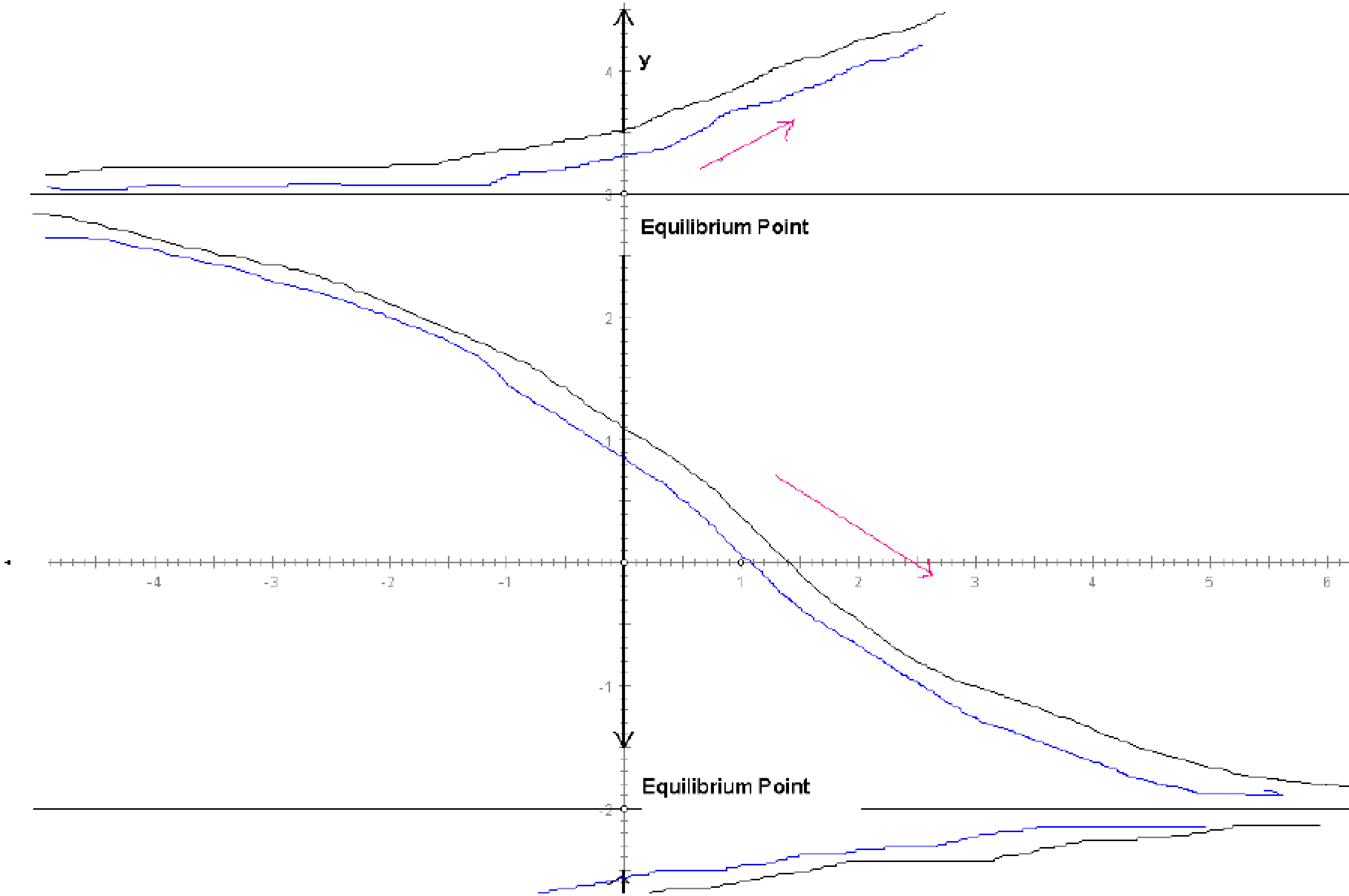
Note that $y^2 - y - 6 > 0$ **for** $(-\infty, -2)$ **and for** $(3, \infty)$
Which means that $y = g(t) \nearrow$ **in these intervals, where** $y = g(t)$ **is a solution**

Note that $y^2 - y - 6 < 0$ **for** $(-2, 3)$
Which means that $y = g(t) \searrow$**in this intervals**

We can illustrate the above graphically using a phase line as shown below


4
3
2
1
y increases
Equilibrium Point
y decreases

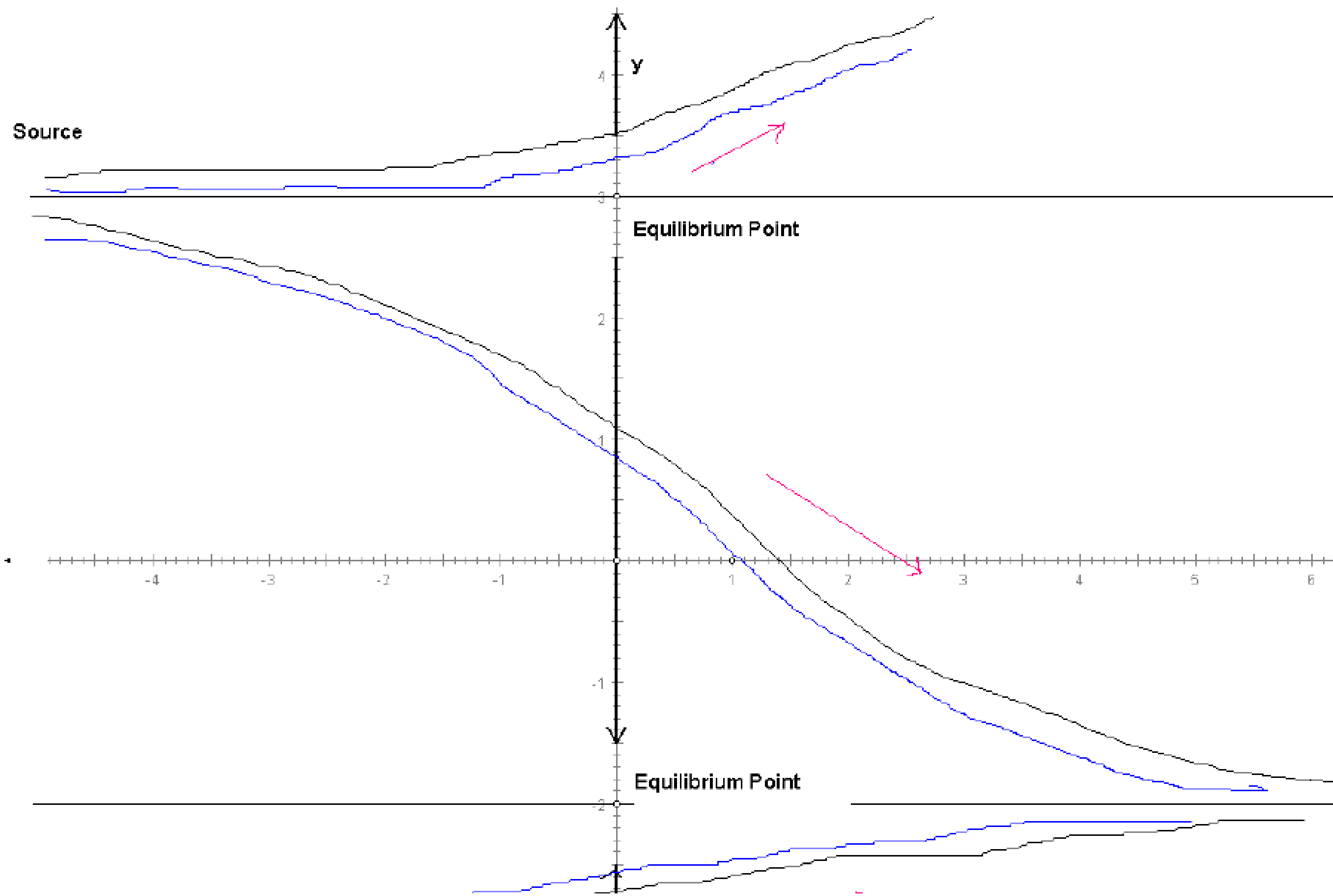
Even though this differential equation can be solved analytically, let us use the above description to sketch a few solution curves



y
4
3
Equilibrium Point
2
1
-4
-3
-2
-1
1
2
3
4
5
6
-1
Equilibrium Point
-2

Such pictures help us designate equilibrium points as source, sink, or nodes

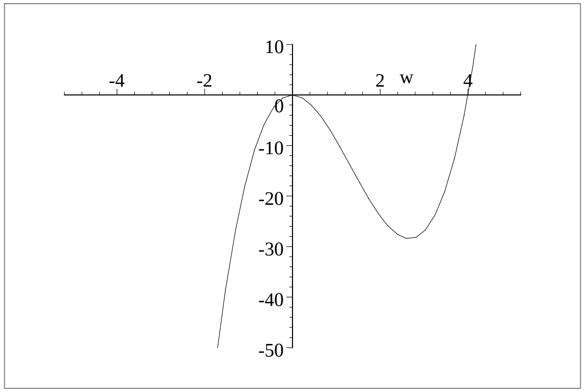
(must look at the pages 86-88, for the definitions and the Linearization Theorem)


y
Source
Equilibrium Point
Equilibrium Point
4
3
2
1
-1
-2
-4
-3
-2
-1
1
2
3
4
5
6

#8 page 91

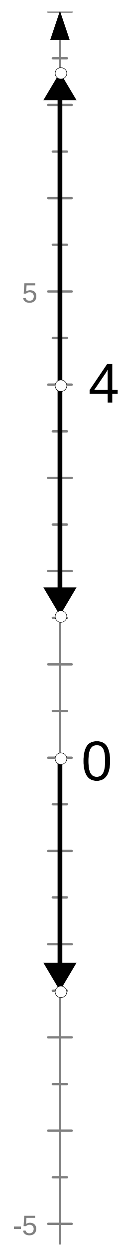
$\frac{dw}{dt} =$ **$3w^3 - 12w^2$**

A graph of $3w^3 - 12w^2$



Equilibrium 0,4

b)


5
4
0
-5

4 is a source
0 is node

#12 on the page 91

$\frac{dw}{dt} = (w^2 - 1)\arctan w$

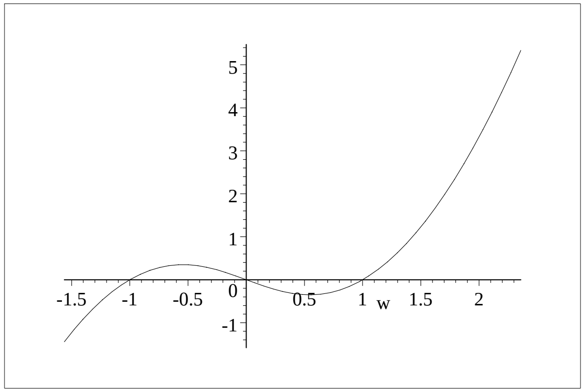
$(w^2 - 1) = 0$ **at** $w = \pm 1$

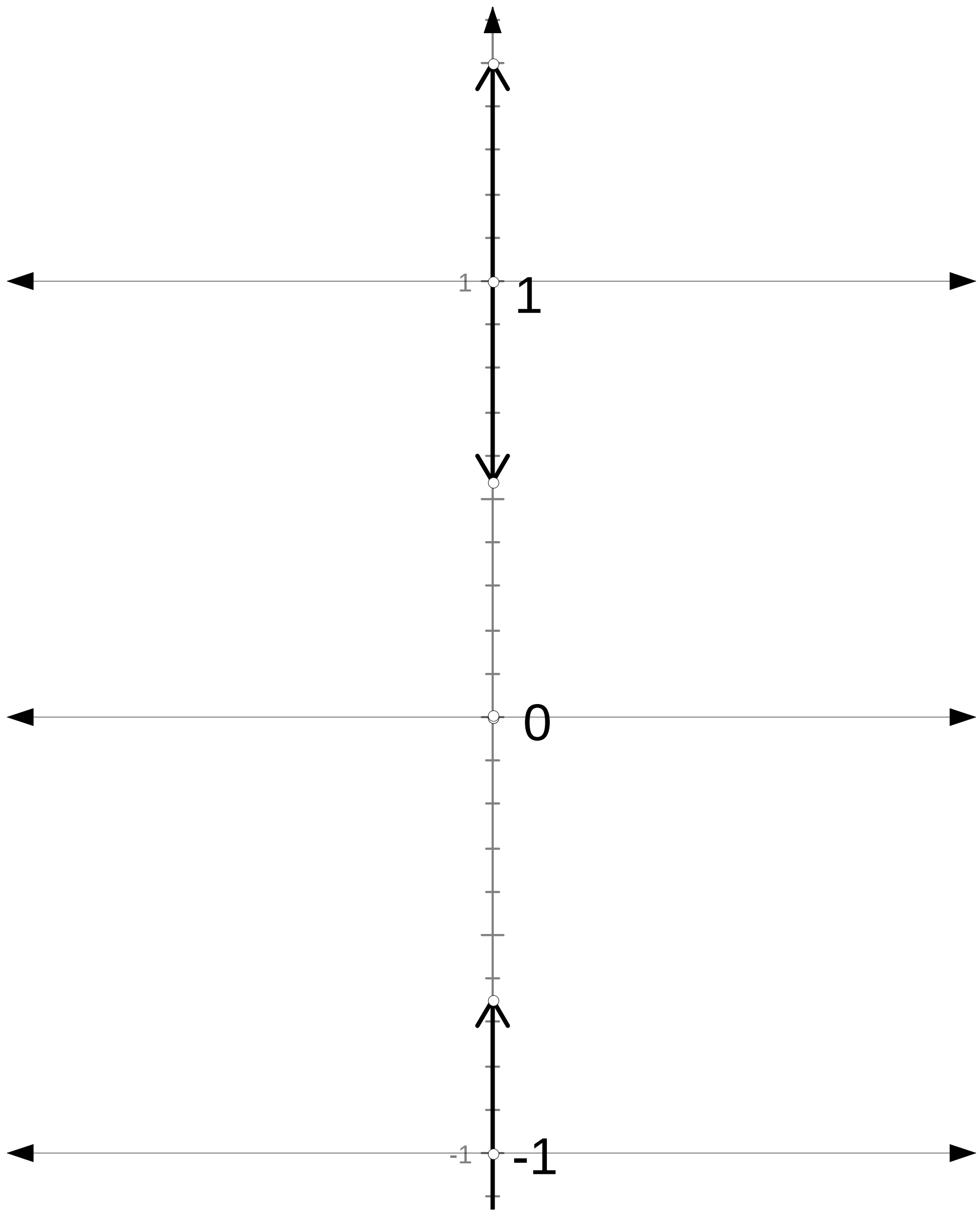
$\arctan w = 0$ **at** $w = 0$

$(w-1)(w+1)\arctan w$

$(-\infty, -1)$	$-$
$(-1, 0)$	$+$
$(0, 1)$	$-$
$(1, \infty)$	$+$

$(w-1)(w+1)\arctan w$




1
1
0
-1
-1

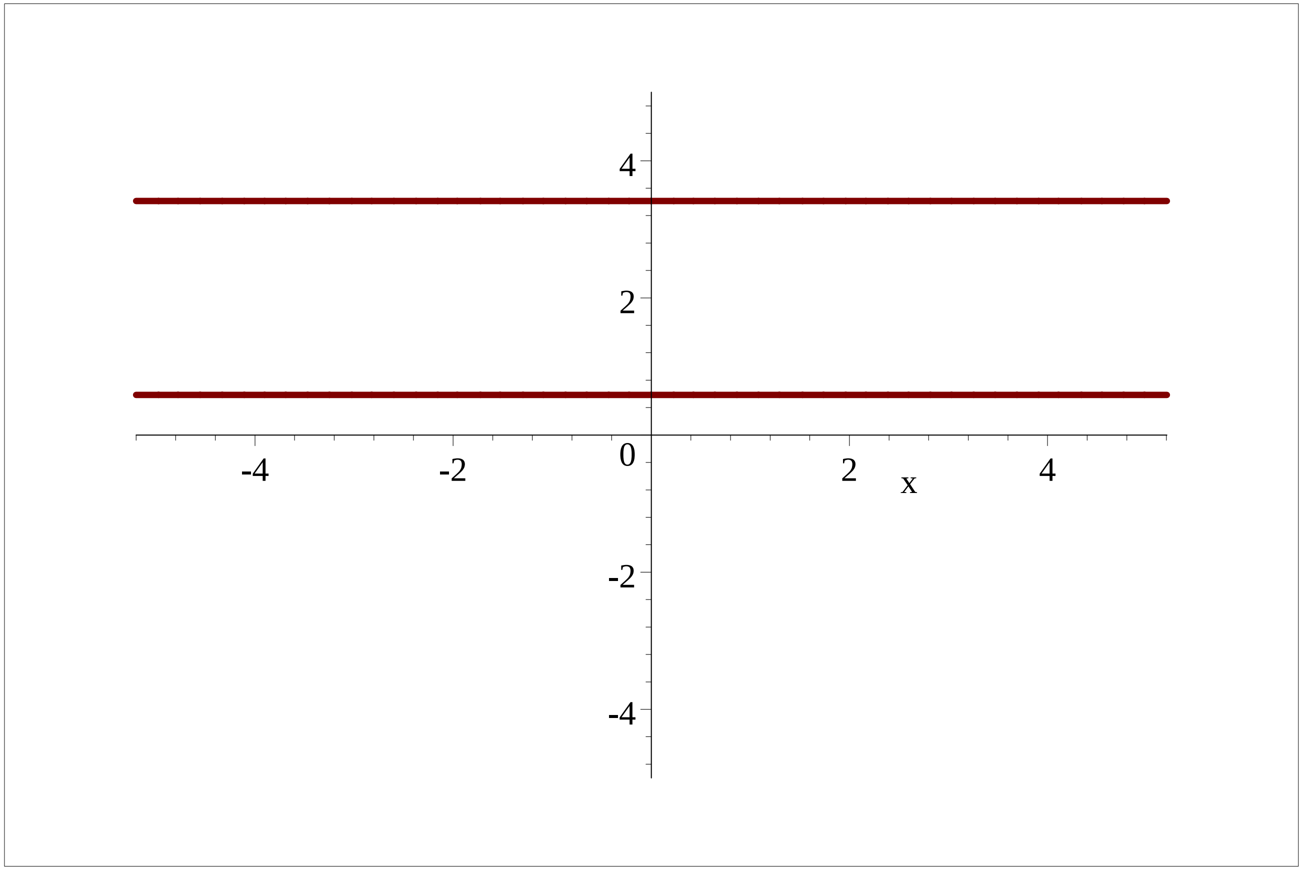
#24 on the page 92

$\frac{dy}{dt} = y^2 - 4y + 2$
$y(0) = -1$

to find the long term behavior

To find the equilibrium points:

$y^2 - 4y + 2 = 0$
$y = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 2}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$
$2 \pm \sqrt{2}$

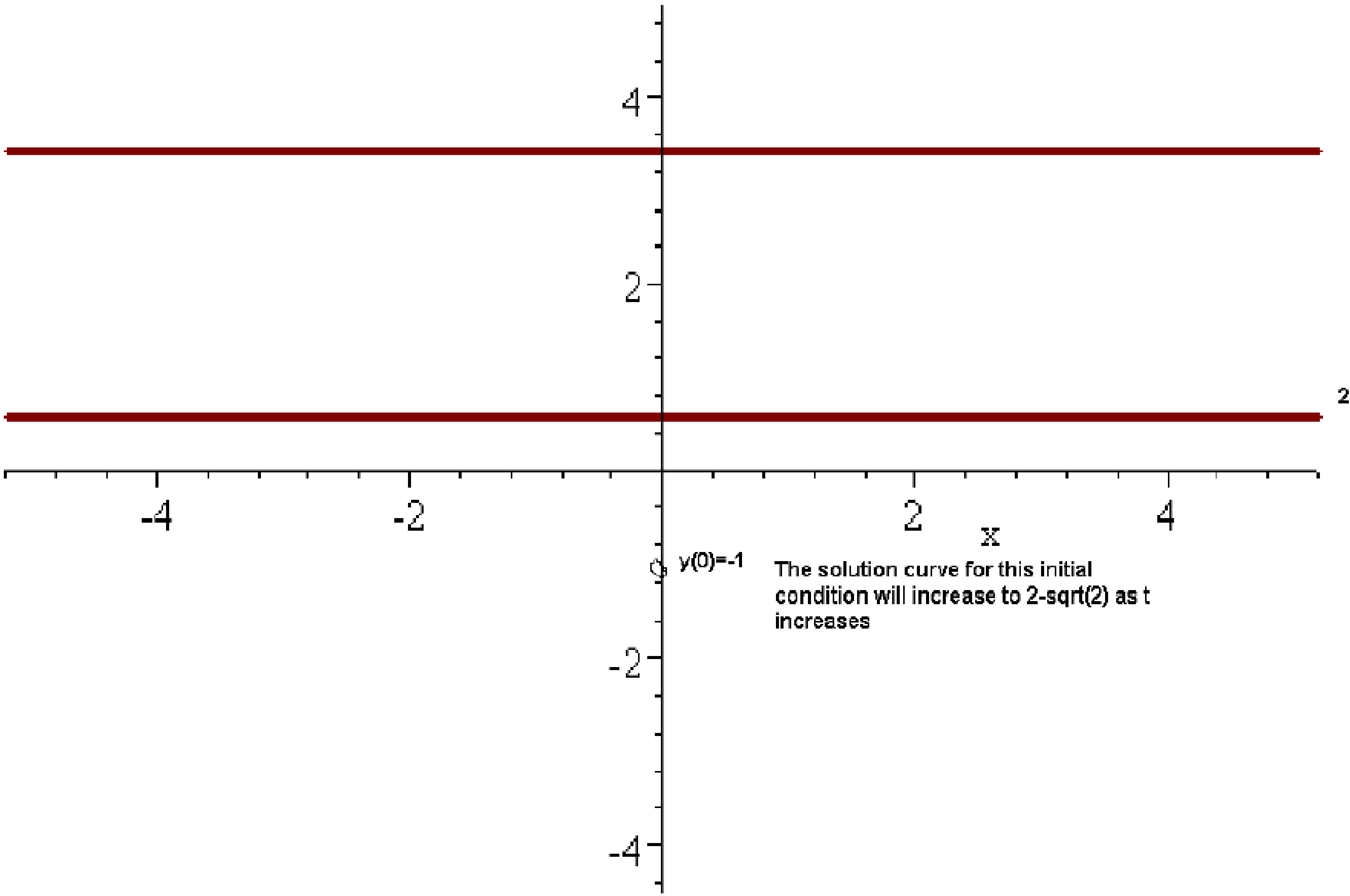


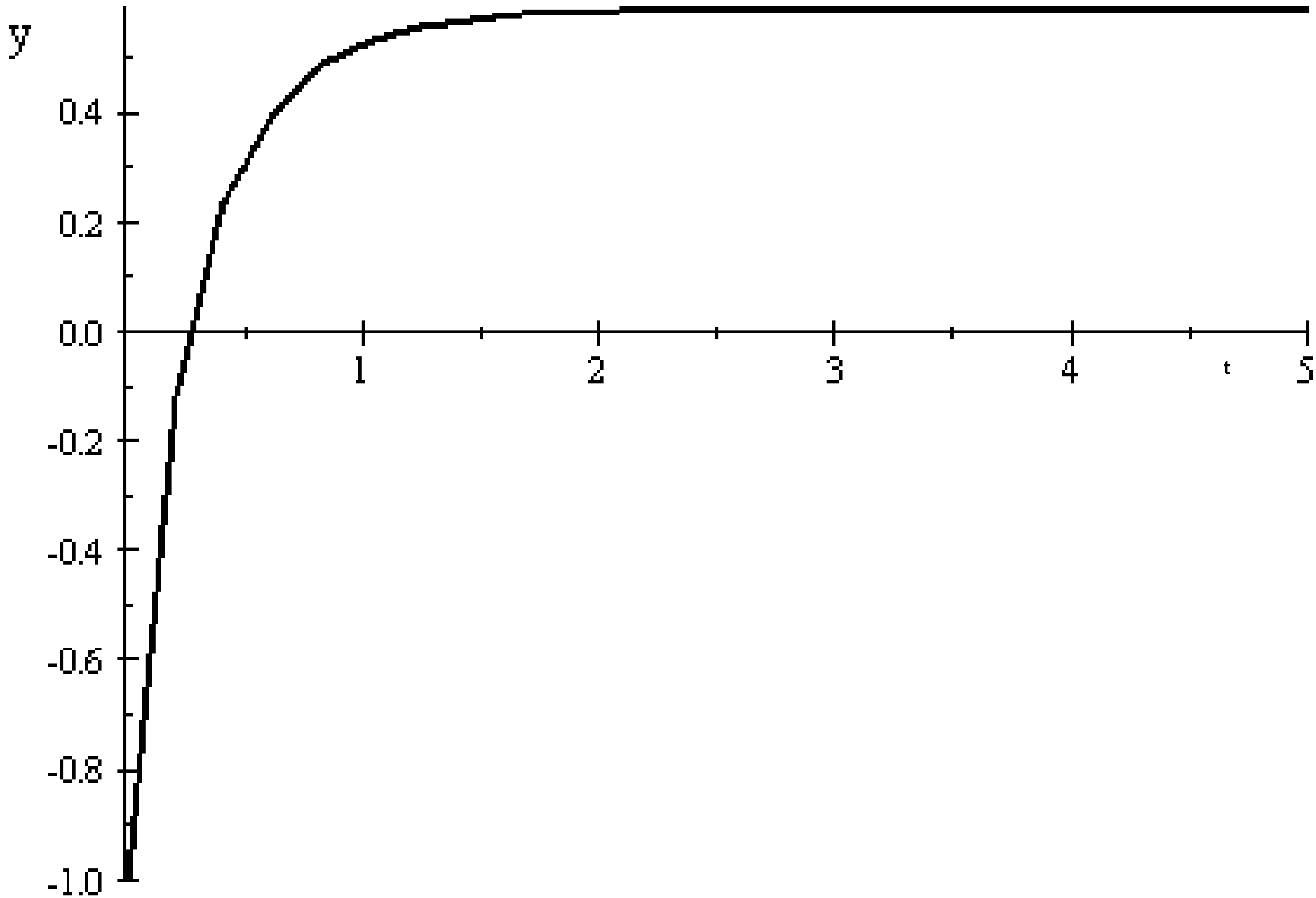
For y in
$\left(-\infty, 2-\sqrt{2}\right)$
$\frac{dy}{dt} = y^2 - 4y + 2$ **is +ve**

as $t \to \infty$**, the solution approaches** $y = 2 - \sqrt{2}$

$\frac{dy}{dt} =$ **$y^2 - 4y + 2$**

$y(0) = -1$


4
2
-2
-4
-4
-2
2
4
x
y(0)=-1
The solution curve for this initial condition will increase to 2-sqrt(2) as t increases
2


y
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
1
2
3
4
t
5

Section 1.8:

Linear Equations
Solve the differential equation

$x^2 \frac{dy}{dx} + 2xy = \sin x$,

$\frac{d(x^2 y)}{dx} = x^2 \frac{dy}{dx} + 2xy$

$x^2 y = w$

$\frac{dw}{dx} = \sin x$

$w = -\cos x + C$

$x^2 y = -\cos x + C$

Exact solution is: $y(x) = \frac{-\cos x + C_1}{x^2}$

Example 2:

To solve

$x^3 \frac{dy}{dx} + x^2 y = 5x^5$

Divide by x^2

$$\mathbf{x}\frac{dy}{dx}+\mathbf{y} = \mathbf{5x}^3$$
$$\frac{d(xy)}{dx} = \mathbf{5x}^3$$

$$\mathbf{xy} = \int \mathbf{5x}^3\mathbf{dx}$$
or
$$\mathbf{xy} = \frac{5x^4}{4}+\mathbf{C}$$

....

Method to solve a linear equation

$$\mathbf{x}\frac{dy}{dx}+\mathbf{y} = \mathbf{5x}^3$$
divide by x
$$\frac{dy}{dx}+\frac{1}{x}\mathbf{y} = \mathbf{5x}^2$$

In general
Have an equation that looks like

$$\frac{dy}{dx}+\mathbf{P}(x)\mathbf{y} = \mathbf{Q}(x)$$

Looking for a function $\mu(x)$ **such that**

$$\boldsymbol{\mu}(x)\frac{dy}{dx}+\boldsymbol{\mu}(x)\mathbf{P}(x)\mathbf{y} = \mathbf{Q}(x)\boldsymbol{\mu}(x)$$

is of the form

$$\frac{d(y\mu(x))}{dx} = \mathbf{Q}(x)\boldsymbol{\mu}(x)$$

Integrate both sides

$$\mathbf{y}\boldsymbol{\mu}(x) = \int \mathbf{Q}(x)\boldsymbol{\mu}(x)\mathbf{dx} + \mathbf{C}$$

How do we obtain $\mu(x)$ **?**

such that

$$\frac{d(y\mu(x))}{dx} = \boldsymbol{\mu}(x)\frac{dy}{dx} + \boldsymbol{\mu}(x)\mathbf{P}(x)\mathbf{y}$$

Note that

$$\frac{d(y\mu(x))}{dx} = \boldsymbol{\mu}(x)\frac{dy}{dx} + \boldsymbol{\mu}'(x)\mathbf{y}$$

$$\boldsymbol{\mu}'(x) = \boldsymbol{\mu}(x)\mathbf{P}(x)$$

$$\frac{d\mu}{dx} = \boldsymbol{\mu}\mathbf{P}$$

$$\frac{d\mu}{\mu} = \mathbf{Pdx}$$

$$\ln \boldsymbol{\mu} = \int \mathbf{Pdx}$$

$$\boldsymbol{\mu} = \mathbf{e}^{\int Pdx}$$

This will give us

$$\mathbf{y}\boldsymbol{\mu}(x) = \int \mathbf{Q}(x)\boldsymbol{\mu}(x)\mathbf{dx} + \mathbf{C}$$

or

$ye^{\int Pdx} = \int Q(x)e^{\int Pdx}dx + C$ **as the solution**

of $\frac{dy}{dx} + P(x)y = Q(x)$

$\frac{dy}{dt} + \mathbf{P}(t)\mathbf{y} = \mathbf{Q}(t)$

Example 1:

To solve

$\frac{dy}{dt} = \frac{2t}{1+t^2}y + \frac{2}{1+t^2}, \quad y(0) = -2$

$\frac{dy}{dt} = \frac{2t}{1+t^2}\mathbf{y} + \frac{2}{1+t^2}$

$\rightarrow$

$\frac{dy}{dt} - \frac{2t}{1+t^2}\mathbf{y} = \frac{2}{1+t^2}$

$\mathbf{e}^{\int Pdt}$

$\mathbf{P}(\mathbf{t}) = -\frac{2t}{1+t^2}$

$\int \mathbf{Pdt} = -\int \frac{2t}{1+t^2}\mathbf{dt}$ $\qquad$ $\mathbf{1} + \mathbf{t}^2 = \mathbf{u} \rightarrow \mathbf{2tdt} = \mathbf{du}$

$\int \mathbf{Pdt} = -\int \frac{du}{u}$

$\int \mathbf{Pdt} = -\ln|u|$

$\int \mathbf{Pdt} = -\ln(1 + t^2)$

$\mathbf{e}^{\int Pdt} = \mathbf{e}^{-\ln(1+t^2)} = \mathbf{e}^{\ln(1+t^2)^{-1}} = \frac{1}{1+t^2}$

Solution is

$\mathbf{y}\frac{1}{1+t^2} = \int \frac{2}{1+t^2}\frac{1}{1+t^2}\mathbf{dt} + \mathbf{C}$

$\mathbf{y}\frac{1}{1+t^2} = \int \frac{2}{(1+t^2)^2}\mathbf{dt}$

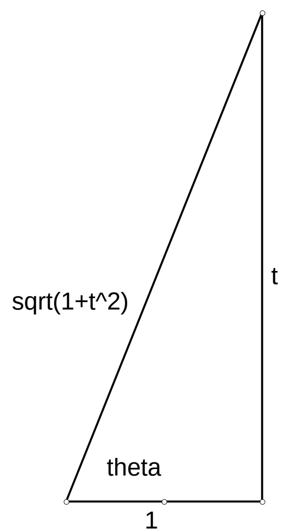
How to obtain

$\int \frac{2}{(1+t^2)^2}\mathbf{dt}$

$\mathbf{t} = \tan\theta \rightarrow \mathbf{dt} = \sec^2\theta \mathbf{d}\theta$

$\mathbf{1} + \mathbf{t}^2 = \mathbf{1} + \tan^2\theta = \sec^2\theta$

$$\int \frac{2}{(1+t^2)^2}\,\mathbf{dt}$$
$$=\int \frac{2}{(\sec^2\theta)^2}\sec^2\theta d\theta$$
$$=2\int \frac{1}{\sec^2\theta}d\theta$$
$$=2\int \cos^2\theta d\theta$$
$$=2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$=\int(1+\cos 2\theta)d\theta$$
$$=\theta+\frac{1}{2}\sin 2\theta$$
$$=\tan^{-1}(t)+\frac{1}{2}2\sin\theta\cos\theta$$
$$=\tan^{-1}(t)+\sin\theta\cos\theta$$

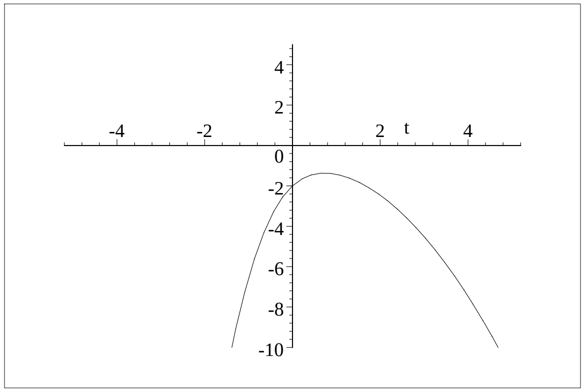


$$\sin\theta=\frac{t}{\sqrt{1+t^2}}$$
$$\cos\theta=\frac{1}{\sqrt{1+t^2}}$$
$$\sin\theta\cos\theta=\frac{t}{1+t^2}$$

$$\int \frac{2}{(1+t^2)^2}\,\mathbf{dt} = \tan^{-1}(t) + \frac{t}{1+t^2}$$

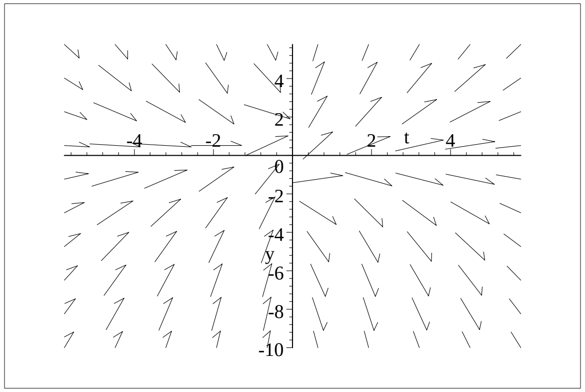
$$\mathbf{y}\frac{1}{1+t^2} = \tan^{-1}(t) + \frac{t}{1+t^2} + \mathbf{C}$$

$$\mathbf{t} = \mathbf{0}, \mathbf{y} = -\mathbf{2}$$
$$(-2)\frac{1}{1+0} = \tan^{-1}(0) + \frac{0}{1+0^2} + \mathbf{C}$$
$$\mathbf{C} = -\mathbf{2}$$
$$\mathbf{y}\frac{1}{1+t^2} = \tan^{-1}(t) + \frac{t}{1+t^2} - \mathbf{2}$$
$$y = (1+t^2)\tan^{-1} t + t - 2(1+t^2)$$



$$\frac{dy}{dt} = \frac{2t}{1+t^2}\mathbf{y} + \frac{2}{1+t^2}$$

$$\left(\frac{1}{\sqrt{1+\left(\frac{2t}{1+t^2}y+\frac{2}{1+t^2}\right)^2}}, \frac{\left(\frac{2t}{1+t^2}y+\frac{2}{1+t^2}\right)}{\sqrt{1+\left(\frac{2t}{1+t^2}y+\frac{2}{1+t^2}\right)^2}} \right)$$



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Example 2:

$\frac{dy}{dt} = -5y + \sin t, \quad y(0) = 1$

$\frac{dy}{dt} + 5y = \sin t$

$e^{\int 5dt} = e^{5t}$

Solution

$ye^{5t} = \int e^{5t} \sin t dt + C$

Use Integration by parts to obtain

$\int e^{5t} \sin t dt = -\frac{1}{26} e^{5t} \cos t + \frac{5}{26} e^{5t} \sin t$

$ye^{5t} = -\frac{1}{26} e^{5t} \cos t + \frac{5}{26} e^{5t} \sin t + C$

$y = -\frac{1}{26} \cos t + \frac{5}{26} \sin t + Ce^{-5t}$

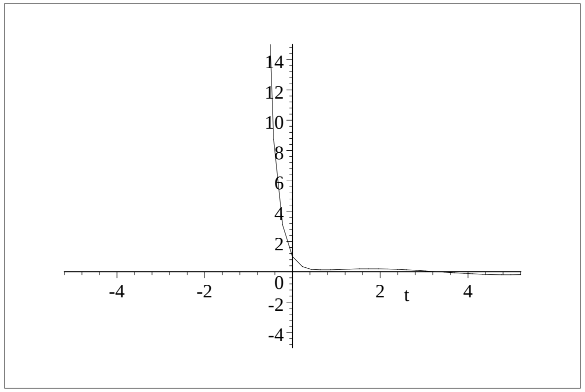
$t = 0, y = 1$

$1 = -\frac{1}{26} \cos 0 + \frac{5}{26} \sin 0 + C$

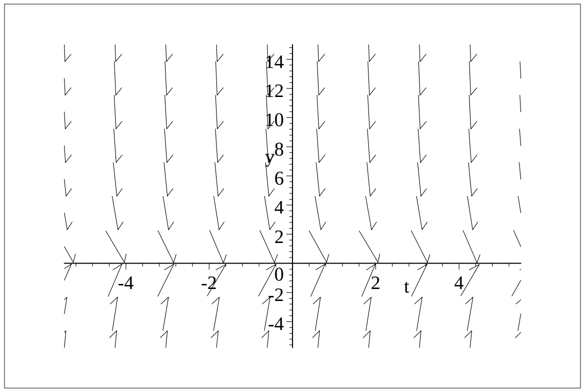
$1 = -\frac{1}{26} + C$

$C = 1 + \frac{1}{26} = \frac{27}{26}$

$$y = -\frac{1}{26}\cos t + \frac{5}{26}\sin t + \frac{27}{26}e^{-5t}$$



$$\left(\frac{1}{\sqrt{1+(-5y+\sin t)^2}}, \frac{-5y+\sin t}{\sqrt{1+(-5y+\sin t)^2}} \right)$$



Example 3:

$$\frac{dy}{dt} + ty = t^3$$

is a linear equation
Multiply by

$\mathbf{e}^{\int t dt} = \mathbf{e}^{t^2/2}$

solution is

$\mathbf{y}\mathbf{e}^{t^2/2} = \int \mathbf{t}^3 \mathbf{e}^{t^2/2} \mathbf{dt}$

$\int \mathbf{t}^3 \mathbf{e}^{t^2/2} \mathbf{dt}$

$\mathbf{t}^2/\mathbf{2} = \mathbf{u} \to \mathbf{tdt} = \mathbf{du}$

$\int \mathbf{t}^3 \mathbf{e}^{t^2/2} \mathbf{dt}$

$= \int t^2 e^{t^2/2} t dt$

$= \int 2ue^u du \quad *$

$= \frac{t^2}{2} \mathbf{e}^{(t^2/2)} - \mathbf{2e}^{(t^2/2)}$

$\mathbf{y}\mathbf{e}^{t^2/2} = \frac{t^2}{2} \mathbf{e}^{(t^2/2)} - \mathbf{2e}^{(t^2/2)} + \mathbf{C}$

$\mathbf{y} = \frac{t^2}{2} - \mathbf{2} + \mathbf{Ce}^{-(t^2/2)}$

$\frac{dy}{dt} + ty = t^3$**, Exact solution is:** $y(t) = t^2 - 2 + e^{-\frac{1}{2}t^2} C_1$

***:**

$\int \mathbf{ue}^u \mathbf{du} = \mathbf{ue}^u - \int \mathbf{e}^u \mathbf{du} = \mathbf{ue}^u - \mathbf{u}$

Example:

$\mathbf{sec}^2 w \frac{dw}{dt} + t \tan w = t$

$\mathbf{y} = \tan \mathbf{w} \to \frac{dy}{dt} = \sec^2 \mathbf{w} \frac{dw}{dt}$

$\mathbf{sec}^2 w \frac{dw}{dt} + t \tan w = t$

becomes

$\frac{dy}{dt} + ty = t$**,**

May treat this as linear or separation of variables
Exact solution is: $y(t) = 1 + e^{-\frac{1}{2}t^2}C_1$

Please work on the sections 1.6, 1.8, and 1.9

If you have difficulty, please post your question in the discussion area